LIE THEORY AND GENERATING RELATIONS OF SPECIAL FUNCTIONS

THESIS SUBMITTED FOR THE DEGREE OF

Doctor of Philosophy

IN

MATHEMATICS

By

ISHRAT JAHAN

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

1987

CERTIFICATE

This is to certify that the contents of this thesis entitled "LIE THEORY AND GENERATING RELATIONS OF SPECIAL FUNCTIONS" is an original research work of Miss Ishrat Jahan, carried under my supervision. She has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University.

I further certify that the work of this thesis has not been submitted to any university or institution, partly or fully, for the award of any other degree.

FORWARDED

(Dr. M. A. Pathan)
Supervisor

CH
Depart ... tics
Aliga ... sity

ACKNOWLEDGEMENT

I am indeed fortunate to have Dr. M. A. Pathan, Reader, Department of Mathematics as my research supervisor. It is a special pleasure to acknowledge my indebtness to him for his meticulous guidance, stimulating encouragement and advice all along.

I wish to record my thanks to Prof. S.I. Husain, Dean, Faculty of Science, Prof. M. Mohsin, former Chairman, and Prof. A.H. Siddiqui, Chairman, Department of Mathematics for providing me all necessary facilities in the department and financial assistance under DSA Scheme.

Finally, my thanks are due to Mr. Masood Alam for typing the thesis very carefully.

Jahan

ALIGARH
November, 1987.

(ISHRAT JAHAN)

SUMMARY OF THE THESIS

The theory of special functions rests on four pillars : analysis, differential equations, topology and algebra. Correspondingly it is possible to distinguish several phases, overlapping in some degree in its development. It also allows us to regard the subject from different points of view and it is the algebraic standpoint which has been chosen in the present thesis. In a monograph by Miller [33], it is shown that the factorization method, a powerful tool for computing eigenvalues and recurrence relations for solutions of second order ordinary differential equations (Infeld and Hull [20]), is equivalent to the representation theory of four local Lie groups. A study of these groups and their Lie algebras leads to a unified treatment of special function theory.

In the course of last two centuries, generating functions of hypergeometric series has received a certain theoretical foundation and has found important applications in the theory of finite differences and algebraic invariants, combinatorial analysis, graph theory, the theory of algorithms and programming on a computer, probability theory, mathematical statistics, analytic number theory, group theory and physics.

The problem of finding the infinite sums (generating functions) of special functions arising in diverse areas of discrete and continuous mathematics, makes up the object of this investigation. The study of this problem is of special interest in the frame work of Lie theory for solving enumeration problems. Interest in the study of special functions by Lie theoretic approach has increased appreciably in recent times. The monographs, books [31], [32], [33], [47], [30], [30], [48], [49] of Miller, Talman McBride, Vilenkin and Wawrzyriczyk are the first devoted entirely to this area of investigation. All this has enabled us to better understand the essence of the proposed tools needed to deduce our results of special functions by obtaining multiplier representations of local Lie groups and representations of Lie algebras by using generalized Lie derivatives. These concepts are introduced in chapter 1 along with a brief survey of classical Lie theory. Most of the definitions stemming from general topology, differentiable manifolds and special functions which are relevent to our work are presented. This serves two purposes. First, it discusses the basic concepts and the background of the Lie groups, Lie algebras and special functions. Second, it seeks to place the study of later chapters in such a way that explicit references may be applied and be brought gradually to a level of considerable understanding. The examples of Local Lie groups presented in this chapter will recur throughout the thesis.

In Chapter 2, we consider a four dimensional complex Lie algebra $\mathcal{G}(a,b)$ with basis J^+, J^-, J^3, ξ. For special values of the parameters a,b, $\mathcal{G}(a,b)$ coincides with one of the Lie algebras $\mathcal{G}(1,o)$, $\mathcal{G}(o,o)$ and $\mathcal{G}(o,1)$. Linear differential operators J^3, J^+, J^- and E of general nature are considered and realizations of the irreducible representions ρ of $\mathcal{G}(a,b)$ such that V becomes a vector space of analytic functions and operators $\rho(\alpha)$, $\alpha \in \mathcal{G}(a,b)$ form a Lie algebra, are obtained. A few known results of Miller [31] Manocha and Jain [23], Jain [22] involving hypergeometric, associated Laguerre and Legendre functions follow as special cases of our findings. We also compute multiplier representation of T_3 and then apply it to obtain a generating function for a special class of polynomials $S_n(x)$ introduced by Schutlz-Piszaebuch [40].

Chapter 3 is devoted to the explicit analysis of the generalized (even and odd) Hermite polynomials theory of the complex local Lie group G (o,1) with four dimensional Lie algebra $\mathcal{G}(o,1)$. The irreducible representation of $\mathcal{G}(o,1)$ is determined by considering the first order linearly independent differential operators J^3, J^+, J^-, E and a realization $\uparrow_{o,2}$ is thus obtained. Generating functions of even and odd generalized Hermite polynomials are obtained from this analysis and a few special cases involving the well known Hermite polynomials are discussed.

In Chapter 4, we determine the scope of our analysis by considering a more general 5-dimensional Lie algebra k_5 which has realizations by generalized Lie derivatives in two complex variables. Corresponding to this Lie algebra, we will obtain generating functions of generalized even and odd Hermite polynomials by relating these polynomials to the representation theory of the Lie algebra. Since the 4-dimensional subalgebra of k_5 generated by J^+, J^-, J^3, $\mathcal{E}$ is isomorphic to $\mathcal{G}(0,1)$, the theory of k_5 is of much wider applicability than the theory of $\mathcal{G}(0,1)$ presented in Chapter 3. Thus this chapter on k_5 is an extension of the Chapter 3 which was devoted to $\mathcal{G}(0,1)$.

In Chapter 5, we establish a fundamental relationship between special linear group SL(2) and a special function which we call as generalized Laguerre-Hermite polynomials $L_{a,b,c,m,n}(x)$. Besides treating the general $L_{a,b,c,m}(x)$ we obtain with the help of a table given in this chapter, a number of special cases involving associated Laguerre, Laguerre Hermite, generalized Hermite, Bessel, Shiveley's pseudo Laguerre, Heat and generalized Heat polynomials and Schultz-Piszachich polynomials. These special cases arise naturally by making use of the results of Szego [46], Rainville [38], Schultz-Piszachich [40], Krall and Frink [27], Haimo [18], Chihara [10] and Bragg [5]. We used the idea of Lie theory as suggested by Miller [33] and Weisner [50], [51], [52] with a view for obtaining generating functions for a class of polynomials $L_{a,b,c,m,n}(x)$. Process involves the introduction of first order linear differential operators generating a Lie algebra isomorphic to sl(2) [33, p. 8] and then based on these operators, we determine a multiplier representation of $[T(g)\ f](x,y)$, $g \in SL(2)$. By choosing the analytic function $f(x,y)$ in different ways, this multiplier representation leads us to generating functions involving the generalized Hermite-Laguerre functions and their products. Special cases lead naturally to many new and known generating functions, the best known of which is the Hille-Hardy formula.

Sections and equations have been numbered chapterwise. A comprehensive bibliography appears at the end with the authors names in alphabetical order. References to the bibliography are numbered in brackets.

CONTENTS

CHAPTER 1

INTRODUCTION

The investigations of S. Lie, aimed at the use of groups in differential equations by studying the structures of the invariant groups of differential operators. It was due to him that the theory of finite continuous groups, known as Lie groups was created. The next stage of the application of group representation theory began with E.P. Wigner who used this theory as a uniform method of studying special functions. His lectures delivered at University of Princeton in 1955 were published in the form of a book in 1968 by J.D. Talman [47]. In the same year there appeared a book by W. Miller [31], who treated by uniform methods a particularly wide class of special functions. A monographic presentation of the theory of special functions by the method of representation theory was also given by N.J. Vilenkin [48] in 1965. Another book by W. Miller [32] (On symmetry groups and their applications) appeared in 1972, who made use of the outstanding achievements of the writings of H. Boerner, I. Gelfand, M. Naimark, N. Vilenkin, H. Weyl and E. Wigner. In 1984, A. Wawrzynczyk [49] published a book Group representations and special functions which differs from Vilenkin's monograph [48] in a sense that a more extensive treatment of the geometric aspects of the theory of special functions is presented.

To make the thesis self contained, in the present chapter we prepare the basic back-ground needed from the theory of Lie-groups, their representations and generating functions of special functions, for the development of the subsequent chapters. Most of the definitions stemming from general topology, manifolds and Lie groups and special functions are presented from [29], [36], [38], [39], [9], [46], (12], [13], [14], [15] and [33].

1.1 Lie Groups

In this section we sketch in brief the idea of Lie groups as well as local Lie groups and discuss some examples which are of relevence to our work.

Definition 1.1.1

A topological space X is said to be locally Euclidean if every point of X has a neighbourhood which is homeomorphic to some open subset in R^n. A locally Euclidean space X is called an n-dimensional topological manifold. In a topological manifold $\forall$ p ϵ X, neighbourhood U and homeomorphism $\phi : U \longrightarrow \phi(U) \subseteq R^n$ form a pair (U, ϕ) called the chart or coordinate pair.

Definition 1.1.2

Let X be an n-dimensional topological manifold and $S = \{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$ be a collection of charts for some index set Λ , the collection S is called a differentiable structure on X if the following conditions hold :

(i) $\bigcup_{\alpha\in\Lambda} U_\alpha = X$,

(ii) whenever $U_\alpha \cap U_\beta \neq \phi$, the maps

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta\ (U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha\ (U_\alpha \cap U_\beta) \text{ and}$$

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha\ (U_\alpha \cap U_\beta) \longrightarrow \phi_\beta\ (U_\alpha \cap U_\beta)$$

are C^∞-maps, that is, partial derivatives of all orders exist and are continuous,

(iii) S is maximal with respect to (i) and (ii).

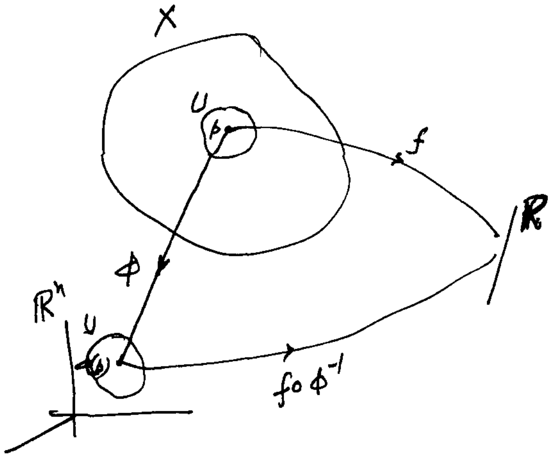
The topological manifold X together with the differentiable structure S is called an n-dimensional differentiable or smooth manifold.

Remark

If in above definition in (ii) "C^∞-maps" are replaced by "analytic maps", then S defines an analytic structure on X and X is called an n-dimensional analytic manifold.

Definition 1.1.3

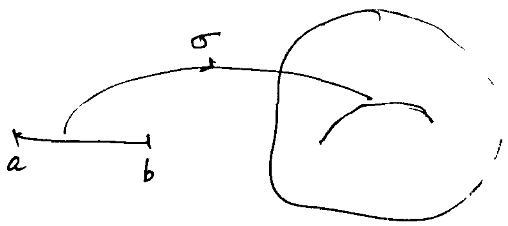
Let X be an n-dimensional smooth (analytic) manifold and let U be a neighbourhood of $p \in X$ such that (U, ϕ) be a chart. A function $f : U \longrightarrow R$ is said to be differentiable (analytic) if $f \circ \phi^{-1} : \phi(U) \longrightarrow R$ is differentiable (analytic) function.



We denote by $C^{\infty}(p)$, the set of all differentiable functions on the neighborhood of $p \in X$.

Definition 1.1.4

Let $I = [a,b]$ be a closed interval and X be an n-dimensional differentiable (analytic) manifold.



A map $\sigma : I \longrightarrow X$ such that $\sigma \circ \phi : U \longrightarrow \phi(U) \subseteq R^n$ is C^{∞}-map (analytic) for every chart (U,ϕ), for which $\sigma \circ \phi$ is defined is called a differentiable (analytic) curve in X. If the manifold is arc-wise connected, through every point we have infinitely many curves.

We associate to each point $p = \sigma(c) \in X$, a linear operator $\alpha : C^{\infty}(p) \longrightarrow R$ by

$$\alpha f = \frac{d}{dt}(f(\sigma(t)))\Big|_{t=c}, \quad f \in C^{\infty}(p),\ t \in I$$

called the tangent vector to the curve σ at $p = \sigma(c)$. The linearity of the operator α follows from the definition, as well as it can also be easily observed that $\alpha(K) = o$ for a constant function K. If the manifold X is arc-wise connected through each point $p \in X$, there pass infinitely many curves and each curve gives rise to a tangent vector α at p. We denote this collection of tangent vectors at p by T_pX called the tangent space at p. We define operations '+' and scalar multiplication '.' on T_pX by

$$(\alpha+\beta)\ f = \alpha f + \beta f \text{ and } (a.\alpha)\ f = a.(\alpha f),$$

$$\alpha,\beta \in T_pX,\ f \in C^{\infty}(p),\ a \in R$$

making T_pX into a vector space over R. Fortunately T_pX turns out to be an n-dimensional vector space (cf. Matsushima [29]).

Now we are ready to introduce Lie-groups and their Lie-algebras.

Definition 1.1.5

An n-dimensional analytic manifold G in which there is given a binary operation '.' with respect to which it is a group also is called a Lie-group if the maps

(i) $(a,b) \in G \times G \longrightarrow a\,b \in G$,

(ii) $a \in G \longrightarrow a^{-1} \in G$

are analytic.

We recall that by an algebra $\mathcal{G}$ we mean a vector space over $R/\mathbb{C}$ in which, in-addition to addition of vectors, the operators .' of vectors is also defined such that $(\mathcal{G}, +, .)$ is a Ring. Also the algebra $\mathcal{G}$ is called a Lie algebra if the multiplication .' satisfies

(i) $\alpha.\alpha = o$, (ii) $\alpha.(\beta.\gamma) + \beta.(\gamma.\alpha) + \gamma.(\alpha.\beta) = o$,

$\alpha,\beta,\gamma \in \mathcal{G}$.

We have seen that tangent space T_pX of a differentiable manifold is a vector space, and therefore T_eG the tangent space of Lie-group G at identity e is also a vector space. Now we demonstrate how the addition structure on G (the group structure) give some additional structure on T_eG. Let $\alpha,\beta \in T_eG$ be the tangent vectors corresponding to the curves $g(t)$ and $h(t)$, $t \in I$, $g(o) = h(o) = e$, $o \in I$. Now using the operation '.' in G and (i), (ii) of definition 1.1.5, we see

$$K(\tau) = g(t)\, h(t)\, g^{-1}(t)\, h^{-1}(t), \quad t = \tau^2$$

defines another analytic curve passing through e. The tangent vector to $K(\tau)$ at e is denoted by $[\alpha,\beta]$. This defines another operation on T_eG called bracket operation and it can be easily seen that this operation satisfies

$$[\alpha,\alpha] = 0, \quad [\alpha, [\beta,\gamma]] + [\beta,[\gamma,\alpha]] + [\gamma, [\alpha,\beta]] = 0,$$

thus T_eG is made into a Lie algebra of a Lie group G.

Definition 1.1.6

A complex abstract Lie algebra $\mathcal{G}$ is a complex vector space together with a multiplication $[\alpha,\beta] \in \mathcal{G}$ defined for all $\alpha,\beta \in \mathcal{G}$ such that

(1) $[\alpha,\beta] = -[\beta,\alpha]$,

(2) $[a_1\alpha_1 + a_2\alpha_2,\beta] = a_1[\alpha_1,\beta] + a_2[\alpha_2,\beta]$, $a_1, a_2 \in \mathbb{C}$,
$\alpha_1, \alpha_2, \beta \in \mathcal{G}$

(3) $[[\alpha,\beta], \gamma] + [[\beta,\gamma],\alpha] = [[\alpha,\gamma],\beta]$, $\alpha, \beta, \gamma \in \mathcal{G}$.

Remark

In theory of Lie-groups, generally Lie algebra is taken to be Lie-algebra of left invariant vector fields, but for our purpose we need only T_eG.

In application of Lie groups (that is Lie theory), many times we do not require global Lie-groups, the one's which we have defined above, but only local Lie groups solve the purpose. These are the groups in which only neighbourhood of identity element carry the differentiable (or analytic) structure compatable to the algebraic structure. Explicitly a local Lie group is defined as follows .

Definition 1.1.7

A complex n-dimensional local Lie group G in the neighbourhood $V \subset \mathbb{C}^n$ is determined by a function $\phi : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that

(1) $\phi(g,h) \in \mathbb{C}^n$ for all $g,h \in V$,

(2) $\phi(g,h)$ is analytic in each of its 2n arguments.

(3) If $\phi(g,h) \in V$, $\phi(h,k) \in V$ then $\phi(\phi(g,h),k) = \phi(g,\phi(h,k))$.

(4) $\phi(e,g) = g$, $\phi(g,e) = g$ for all $g \in V$.

Remark

In general theory of Lie groups as we have already pointed out that generally its Lie-algebra consists of left invariant vector fields, and there is an association of this Lie algebra to the Lie group by a celebrated map "exp." called the exponential map, and is a local diffeomorphism. This notion carryover naturally to our Lie algebra also, namely, $\exp : T_eG \longrightarrow G$. Thus $\forall\ \alpha \in T_eG$, $\exp \alpha \in G$. This is well demonstrated by the following fact that for $t \in R$, $\exp t\alpha$ gives rise to a curve in G, whose tangent vector at e is α. It is also obvious that $\exp o = e$, where o is zero tangent vector.

It is worth mentioning that in case of Lie groups of matrices, this exponential map coincides with usual exponential of matrices (cf. examples given below). Further exponential maps being local diffeomorphisms, the notion extends to local Lie groups also.

Examples

We list some examples of global (local) Lie groups which will be subsequently used in our work.

1. The general linear group $GL(n, \mathbb{C})$

The set of all non-singular n x n complex matrices denoted by $GL(n, \mathbb{C})$ is a group with respect to usual multiplication of matrices. The determinant function $d : GL(n, \mathbb{C}) \longrightarrow \mathbb{C}$, equipps $GL(n, \mathbb{C})$ with a topology, with respect to which $GL(n, \mathbb{C})$ becomes an open set of $\mathbb{C}^{n^2}$

($\mathbb{C}^{n^2}$ being identified with the set of all n x n matrices over $\mathbb{C}$ by writing a matrix as an n^2-tuple of complex numbers). From definition of manifolds it trivially follows that open sets of an n-dimensional manifolds are n-dimensional manifolds. Moreover C^{n^2} is automatically equipped with analytic structure containing single chart (C^{n^2}, I) and therefore an analytic manifold. With these considerations GL(n, $\mathbb{C}$) becomes an n^2-dimensional analytic manifold (therefore $2n^2$ real). It is easy to see that group operation and inversion are analytic functions and therefore GL(n, $\mathbb{C}$) is an n^2-dimensional Lie group. We remark that the Lie-algebra of the Lie-group GL(n, $\mathbb{C}$) is the set of all n x n matrices and therefore $\mathbb{C}^{n^2}$.

2. <u>The special linear group SL(2)</u>

The set of all non-singular 2 x 2 complex matrices

$$g = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{C} \right\},$$

such that det g = 1, is denoted by SL(2). It is certainly a subgroup of GL(2, C), Furthermore, SL(2) is 3-dimensional analytic Lie group. Let us compute its Lie algebra which is denoted by sl(2). Clearly for every A $\in$ sl(2) exp A = $e^A \in$ SL(2) and therefore det (e^A) = 1 gives us trace A = o. Thus sl(2) =

{all 2 x 2 matrices over $\mathbb{C}$ with tr. A = o}. One can choose the following basis for sl(2), generally used in applications

$$e^{+} = \begin{pmatrix} o & -1 \\ o & o \end{pmatrix}, \quad e^{-} = \begin{pmatrix} o & o \\ -1 & o \end{pmatrix}, \quad e^{3} = \frac{1}{2}\begin{pmatrix} 1 & o \\ o & -1 \end{pmatrix},$$

which satisfies the following relations

$$[e^{+}, e^{-}] = 2e^{3}, \; [e^{3}, e^{\pm}] = \pm e^{\pm}.$$

3. $\underline{T_3}$

The set of all matrices of the form

$$\begin{pmatrix} 1 & o & o & \tau \\ o & e^{-\tau} & o & c \\ o & o & e^{\tau} & b \\ o & o & o & o \end{pmatrix}, \quad b, c, \tau \in \mathbb{C}$$

is denoted by T_3. It is a 3-dimensional simply connected local Lie group (cf. [31], p. 11). The Lie-algebra of T_3 is denoted by τ_3 and it consists of the space of matrices of the form

$$\begin{pmatrix} o & o & o & x_3 \\ o & -x_3 & o & x_2 \\ o & o & x_3 & x_1 \\ o & o & o & o \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{C}.$$

A basis of τ_3 can be chosen as

$$e^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which satisfies the following relations

$$[e^3, e^+] = e^+, \; [e^3, e^-] = -e^-, \; [e^+, e^-] = \mathcal{O}.$$

4. G(0,1)

The set of all matrices of the form

$$\begin{pmatrix} 1 & ce^\tau & a & \tau \\ 0 & e^\tau & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, c, \tau \in \mathbb{C}$$

is denoted by $G(0,1)$. It is a 4-dimensional Lie group (cf. [31], p. 9). The Lie algebra of $G(0,1)$ is denoted by

L(G(o,1)) and it consists of the space of matrices of the form

$$\begin{pmatrix} 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad x_1, x_2, x_3, x_4 \in \mathbb{C}.$$

A basis of L(G(o,1)) can be chosen as

$$j^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad j^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$j^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. $\underline{K_5}$

The set of all 5 x 5 matrices of the form

$$g(q,a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & be^{-\tau} & 2a-bc & \tau \\ 0 & e^{\tau} & 2qe^{-\tau} & b-2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, a,b,c,q,\tau \in \mathbb{C}$$

is denoted by K_5. It is a 5-dimensional Lie group. Its Lie algebra is isomorphic to the 5-dimensional Lie algebra k_5 having basis elements e^3, e^+, e^-, ε, $\mathcal{J}$ satisfying the commutation relations :

$$[e^3, e^{\pm}] = \pm e^{\pm}, \ [e^3, \mathcal{J}] = 2\mathcal{J}, \ [e^-, e^+] = \varepsilon,$$

$$[e^-, \mathcal{J}] = 2e^+, \ [e^+, \mathcal{J}] = 0, \ [e^{\pm}, \varepsilon] = [e^3, \varepsilon] = [\mathcal{J}, \varepsilon] = 0$$

(cf. [31], p. 299).

1.2 Representation Theory

The primary tools needed to deduce our results in special functions are representation of Lie groups and representation of Lie algebras by generalized Lie derivatives. Special functions occur as matrix elements and basis vectors corresponding to representation of local Lie groups. We shall be interested only in a special kind of representation called the multiplier representation. We recall that given a group G and a vector space V, by representation of G in V we mean a group homomorphism $T : G \longrightarrow GL(V)$, where $GL(V)$ is a group of non-singular linear transformations of V. Thus $\forall\, g \in G$, we have a linear transformation $T_g : V \longrightarrow V$. A similar definition applied to the representation of Lie group G into V. Before defining multiplier representation we shall first discuss action of a Lie group on a manifold.

Definition 1.2.1

Let G be an analytic Lie group and M be an analytic manifold. G is said to act on M if

(i) the map $M \times G \longrightarrow M$, $(m, g) \longrightarrow mg$ is analytic, $m \in M$, $g \in G$.

(ii) $m.e = m$; e is identity of G.

(iii) $(xg_1).g_2 = x(g_1.g_2)$, $g_1, g_2 \in G$.

When G acts on M it is also called as Lie group of transformation of M.

Let U be an open sub-set of C^n, the n-dimensional complex space such that $o \in U$. Then U is an analytic sub-manifold of C^n. Suppose a Lie group G acts on U, then we shall see that the action of G on U induces action of G on the space of all analytic functions defined on U as follows :

Let $\mathcal{G}$ be the Lie-algebra of the Lie group G, then we know for $\alpha \in \mathcal{G}$, expt α is a curve in G whose tangent vector at $t = o$ is α. For an analytic function f defined on U, we define an analytic function (exptα) f by

$$[(\exp t\alpha)f](x) = f(x\exp t\alpha), \quad x \in U, \ \alpha \in \mathcal{G}.$$

Here we remark that if G is a global Lie-group, the above definition works for all $t \in R$, however, if G is a local Lie-group, then we have to take $|t|$ small enough so that the analyticity of f is not lost.

Now we define the multiplier representation :

<u>Definition 1.2.2</u> (Miller [30], p. 17)

Let G be a local Lie-group acting on an open neighbourhood U of o in $\mathbb{C}^n$, and let cl be the set of all complex valued analytic functions of U. A multiplier representation T^ν of G on cl with multiplier ν, consists of a mapping $T^\nu(g)$ of cl onto cl defined for all $g \in G$, $f \in$ cl by

$$[T^\nu(g)f](x) = \nu(x,g) \ f(xg), \quad x \in U,$$

where $\nu : U \times G \longrightarrow \mathbb{C}$ is an analytic function and satisfies

(i) $\quad \nu(x,e) = 1, \quad$ all $x \in U$,

(ii) $\quad \nu(x,g_1g_2) = \nu(x,g_1)\ \nu(xg_1, g_2), \ g_1, g_2, g_1g_2 \in G.$

Property (2) is equivalent to the relation

$$[T^\nu(g_1g_2)f](x) = [T^\nu(g_1)(T^\nu(g_2)f)](x).$$

Definition 1.2.3

The generalized Lie derivative $D_\alpha f$ of an analytic function $f(x)$ under the 1-parameter group $\exp t\alpha$ is the analytic function

$$(D_\alpha f)(x) = \frac{d}{dt}\left[\left(T^{\nu} (\exp t\alpha)\, f\right] (x)\right|_{t = 0}.$$

For $\nu = 1$ the generalized Lie derivative becomes the ordinary Lie derivative.

The generalized Lie derivative of a local Lie transformation group form a Lie algebra under the operations of addition of derivatives and Lie product

$$[D_\alpha, D_\beta] = D_\alpha D_\beta - D_\beta D_\alpha .$$

This algebra is a homomorphic image of $L(G)$:

$$D_{\alpha+\beta} = D_\alpha + D_\beta,\; D_{[\alpha,\beta]} = [D_\alpha, D_\beta],\; D_{a\alpha} = aD_\alpha.$$

Now we state the following theorem which is of fundamental importance in the Lie-theory and special functions (cf. [31], p. 18) in a little modified form for our use.

<u>Theorem 1.2.1</u>

Let

$$D_j(x) = \sum_{i=1}^{m} P_{ji}(x)\, \partial/\partial x^i + P_j(x),\ j = 1, \ldots, n$$

be n-linearly independent differential operators defined and analytic in an open set $U \subset \mathbb{C}^n$. If there exist constants C_{jk}^{ℓ} such that

$$[D_j, D_k] = D_j D_k - D_k D_j = \sum_{\ell=1}^{n} C_{jk}^{\ell} D_\ell, \qquad \ell \leq j, k < n,$$

then the complex linear combinations of D_j form a Lie algebra which is the algebra of generalized Lie derivatives of an effective local multiplier representation T^{ν}. The action of the group G is obtained by the integration of the equations

$$\frac{d}{dt} x_i(t) = \sum_{j=1}^{n} \alpha_j P_{ji}(x(t)),\ x_i(o) = x_i^o\ ;\ i = 1, 2, \ldots, m$$

$$\frac{d}{dt} \nu(x^o, \exp\alpha t) = \nu(x^o, \exp\alpha t) \sum_{j=1}^{n} \alpha_j P_j\,(x(t)),\ \nu(x^o, e) = 1.$$

where $x(t) = x^o \exp\alpha t$, $\alpha \in L(G)$.

1.3 <u>Representation of $\mathcal{G}(a,b)$ and the representation $R(\omega, m_o, \mu)$</u>

In this section we consider the Lie algebra $\mathcal{G}(a,b)$

which for every pair of complex numbers (a,b) is a 4-dimensional complex Lie algebra generated by the basis elements e^+, e^-, e^3 and I satisfying :

$$[e^+, e^-] = 2a^2e^3 - bI, \ [e^3, e^+] = e^+, \ [e^3, e^-] = -e^-,$$

$$[e^+, I] = [e^-, I] = [e^3, I] = o \ , \qquad (1.3.1)$$

where o is the additive identity, and study its representation on a complex vector space.

Let ρ be a representation of $\mathcal{G}(a,b)$ on a complex vector space V and put

$$J^+ = \rho(e^+), \ J^- = \rho(e^-), \ J^3 = \rho(e^3) \ , \ E = \rho(I). \quad (1.3.2)$$

Then ρ being Lie algebra representation, the operators J^+, J^-, J^3, E obey the same relations as (1.3.1). Let S be the spectrum of the operator J^3. Then the multiplicity of the eigenvalue $\lambda \in S$ is the dimension of the eigenspace $V^3 = \{v \in V | J^3v = \lambda v\}$.

We shall analyse the irreducible representation of $\mathcal{G}(a,b)$ and for each such representation we find a basis of V consisting of eigen vectors of J^3, that is, we shall classify all representations ρ of $\mathcal{G}(a,b)$ satisfying

(i) ρ is irreducible.

(ii) Each eigenvalue of J^3 has multiplicity one. (1.3.3)

The basic justification for the above requirements is that they quckly lead to the connection between $\mathcal{G}(a,b)$ and certain special functions.

Here our object is to test all the possibilities of ρ. For this, first we remark that

(A) Define the operator $C_{a,b}$ on V by

$$C_{a,b} = J^+J^- + a^2J^3J^3 - a^2J^3 - bJ^3E.$$

It is easy to see that this operator commutes with J^+, J^-, J^3 and E and that $C_{a,b} = \lambda I_d$ where I_d is identity operator and λ is a complex number which depend on ρ.

(B) The spectrum S is connected subset of $\mathbb{C}$.

(C) The representation ρ of $\mathcal{G}(a,b)$ is uniquely determined by λ, μ and the spectrum of S.

The proof of above observations can be found in (cf. [31], pp. 40-41).

We need only to consider the Lie algebra $\mathcal{G}(o,o)$ $\mathcal{G}(o,1)$ and $\mathcal{G}(1,o)$ since $\mathcal{G}(a,b)$ is isomorphic to one of the three. We have

Theorem 1.3.1

Every representation of $\mathcal{G}(o,o)$ which satisfies (1.3.3) and for which $J^+J^- \neq o$ on V is isomorphic to a representation $Q^{\mu}(w, m_o)$ defined for μ, w, $m_o \in \mathbb{C}$ such that $w \neq o$ and $o \leq \text{Re } m_o < 1$. $S = \{m_o + n \mid n \text{ an integer}\}$. For each representation $Q^{\mu}(w, m_o)$ there is a basis for V consisting of vectors f_m, $m \in S$, such that

$$J^3 f_m = m\, f_m,\ E\, f_m = \mu\, f_m,$$

$$J^+ f_m = w\, f_{m+1},\ J^- f_m = w\, f_{m-1},$$

$$C_{o,o}\, f_m = J^+J^-\, f_m = w^2\, f_m.$$

Theorem 1.3.2

Every representation of $\mathcal{G}(o,1)$ satisfying (1.3.3) and for which $E \neq o$ is isomorphic to representation in the following list :

(i) The representations $R(w, m_o, \mu)$ defined for all w, m_o, $\mu \in \mathbb{C}$ such that $\mu \neq o$, $o \leq \text{Re } m_o < 1$ and $w + m_o$ is not an integer. $S = \{m_o + n \mid n \text{ an integer}\}$.

(ii) The representations $\uparrow_{w,\mu}$ defined for all w, $\mu \in \mathbb{C}$ such that $\mu \neq o$. $S = \{-w + n \mid n \text{ a non-negative integer}\}$.

For each cases (i) and (ii) there is a basis of V consisting of vectors f_m defined for each $m \in S$ such that

$$J^3 f_m = m\, f_m,\ E\, f_m = \mu\, f_m,$$

$$J^+ f_m = \mu\, f_{m+1},\ J^- f_m = (m+w)\, f_{m-1},$$

$$C_{o,1}\, f_m = (J^+J^- - E\, J^3)\, f_m = \mu\, w\, f_m.$$

(iii) The representation $\downarrow_{w,\mu}$ defined for all $w, \mu \in \mathbb{C}$ such that $\mu \neq o$. $S = \{-w-n \mid n$ a non-negative integer$\}$. For each of the representation there is a basis of V consisting of vectors f_m defined for each $m \in S$ such that

$$J^3 f_m = m\, f_m,\ E\, f_m = -\mu f_m,$$

$$J^+ f_m = -(m+w+1)\, f_{m+1},\ J^- f_m = \mu\, f_{m-1},$$

$$C_{o,1}\, f_m = (J^+J^- - E\, J^3)\, f_m = -\mu\, w\, f_m.$$

Theorem 1.3.3

Every representation ρ of $\mathcal{G}(1,o)$ satisfying the conditions (1.3.3) is isomorphic to a representation in the following list :

(i) The representations $D^{\mu}(u, m_o)$ defined for all complex μ, u, m_o such that m_o+u, m_o-u are not integers and $o \leq \mathrm{Re}\, m_o < 1$. $S = \{m_o+n \mid n$ an integer$\}$. $D^{\mu}(u, m_o)$ and $D^{\mu}(-u-1, m_o)$ are isomorphic.

(ii) The representations $\uparrow^{\mu}_{u}$, μ, $u \in C$, where 2 u is not a non-negative integer. $S = \{-u + n : n$ a non-negative integer$\}$.

(iii) The representations $\uparrow^{\mu}_{u}$, μ, $u \in \mathbb{C}$, where 2 u is not a non-negative integer. $S = \{u - n : n$ a non-negative integer$\}$.

(iv) The representations $U^{\mu}(2u)$ where 2 u is a non-negative integer. $S = \{u, u-1, \ldots, -u+1, -u\}$.

For each of these representations there is a basis of V consisting of vectors f_m defined for each $m \in S$ such that

$$J^3 f_m = m\, f_m,\ J^+ f_m = (m-u)\, f_{m+1},$$

$$J^- f_m = -(m+u)\, f_{m-1},\ E\, f_m = \mu\, f_m,$$

$$C_{1,o}\, f_m = (J^+J^- + J^3J^3 - J^3)\, f_m = u(u+1)\, f_m.$$

We have seen in Theorem (1.3.2) that the irreducible representation $R(w, m_o, \mu)$ of $\mathcal{G}(o,1)$ is determined by complex constants w, m_o, μ such that $\mu \neq o$, $o \leq R_e\, m_o < 1$ and $w + m_o$ is not an integer. Here spectrum S is given by

$$S = \{m \ + n : n \text{ an integer}\}$$

and the representation space V has basis $\{f_m\}$, $m \in S$ such that

$$J^3 f_m = m\, f_m,\ E\, f_m = \mu\, f_m,\ J^+ f_m = \mu\, f_{m+1},$$

$$J^- f_m = (m+w)\, f_{m-1},\ C_{o,1} f_m = (J^+J^- - E\, J^3) f_m = \mu\, w\, f_m \tag{1.3.4}$$

and the commutation relations are given by

$$[J^3, J^{\pm}] = \pm J^{\pm}, \ [J^+, J^-] = -E, \qquad (1.3.5)$$
$$[J^{\pm}, E] = [J^3, E] = 0.$$

We conclude this section by stating.

<u>Theorem 1.3.4</u>

Every irreducible representation ρ of K_5 satisfying property A, $\rho / \mathcal{G}(0,1)$ (ρ restricted to the subalgebra $\mathcal{G}(0,1)$) is isomorphic to one of the irreducible representations $R(w, m_0, \mu)$ or $\uparrow_{w,\mu}$ of $\mathcal{G}(0,1)$, is isomorphic to a representation in the following list :

(i) $R'(w,m_0,\mu)$ $w, m_0, \mu \in \mathbb{C}, \mu \neq 0,$

$0 \leq R_e(m_0) < 1$, $w+m_0$ not an integer,

the spectrum of J^3 is the set $S = \{m_0 + n : n \text{ an integer}\}$,

(ii) $\uparrow'_{w,\mu}$ $w, \mu \in \mathbb{C}, \ \mu \neq 0.$

The spectrum of J^3 is the set $S = [-w+n :$ nonnegative integer]

For each of the above classes the representation space V has a basis $\{f_m\}$, $m \in S$, such that

$$J^3 f_m = m f_m, \ E f_m = \mu f_m, \ Q f_m = \mu f_{m+2},$$

$$J^+f_m = \mu f_{m+1}, \quad J^-f_m = (m+w) f_{m-1}$$

for all $m \in S$ on the left hand sides of these equations. All of the irreducible representations in classes (i) and (ii) satisfy property and no two of them are isomorphic.

1.4 Generating function

If a sequence of numbers $g_1, g_2, \ldots.$ is determined as the sequence of coefficients in the expansion into an infinite series of a certain function, then this function is called the generating function of g_n.

The most frequent type of infinite series in this connection is a power series.

$$G(x_1, x_2, \ldots. x_p; t) = \sum_{n=0}^{\infty} C_n g_n (x_1, \ldots x_p) t^n, \tag{1.4.1}$$

where C_n is a specified sequence, independent of variables and t. G is called the generating function of $g_n (x_1, x_2, \ldots, x_p)$ and $x_1, x_2, \ldots. x_p$, t are regarded as p+1 independent variables.

As a rule, the power series occurring as generating function has a positive radius of convergence. Sometimes, however, it is useful also to consider power series which have zero radius of convergence, that is to say are divergent except for t = o.

The name generating function was introduced by Laplace in 1812.

1.5 Hypergeometric function

Making use of the Pochhammer symbol $(a)_n$ defined as

$$(a)_n = (a,n) = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\ldots(a+n-1), & \text{if } n = 1, 2, \ldots \end{cases}, \quad (1.5.1)$$

the generalized hypergeometric function is given by

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix} Z\right] = \sum_{n=0}^{\infty} \frac{\left[\prod_{i=1}^{p}(a_i)_n\right] Z^n}{\left[\prod_{i=1}^{q}(b_i)_n\right] n!} \quad (1.5.2)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n, \ldots, (a_p)_n Z^n}{(b_1)_n, \ldots, (b_q)_n n!},$$

where $(a_i)_n = \dfrac{\Gamma(a_i+n)}{\Gamma(a_i)}$,

and where the numerator parameters $a_1, a_2, \ldots a_p$ may be zero or negative but the denominator parameters $b_1, b_2, \ldots, b_q$ are not allowed to be zero or negative.

The convergence conditions of ${}_pF_q$ investigated by means of D'Alembert ratio test are as mentioned below :

(i) If $p \leq q$, the series converges for all finite Z (real or complex) and diverges absolutely when $Z = 1$.

(ii) If $p = q + 1$, the series converges for $|Z| < 1$ and diverges for $|Z| > 1$.

(iii) If $p > q+1$, the series converges only when $Z = o$ and diverges when $Z \neq o$.

(iv) If $p = q+1$, the series is absolutely convergent on the circle $|Z| = 1$, i.e.,

$$R_e \left(\sum_{j=o}^{q} b_j - \sum_{i=1}^{p} a_i \right) > o \quad \text{for } Z = 1$$

and

$$R_e \left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \right) > o \quad \text{for } Z = -1.$$

Hypergeometric forms of some special functions

(i) When $p = q = 1$, (1.4.2) reduces to the confluent hypergeometric function ${}_1F_1$ named as Kummer's function given by E.E. Kummer in 1836 :

$${}_1F_1\,(a,c;Z) = \sum \frac{(a)_n\, Z^n}{(c)_n\, n!} \quad . \tag{1.5.3}$$

(ii) When $p = 2$ and $q = 1$, (1.4.2) reduces to an ordinary hypergeometric function of second order ${}_2F_1$ and was given by C.F. Gauss in the year 1812 :

$$ {}_2F_1\,(a,b;c;Z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{Z^n}{n!} \;. \qquad (1.5.4)$$

(iii) If p = o, q = o, i.e., if no numerator or dinominator parameters are present, the equation (1.4.2) reduces to the exponential function

$$ {}_0F_0\,(-\,;\,-\,;\,Z) = \sum_{n=0}^{\infty} \frac{Z^n}{n!} = \exp\,(Z) \;. \qquad (1.5.5)$$

(iv) If p = 1 and q = o, (1.4.2) gives the binomial function :

$$ {}_1F_0\,(a\,;\,-\,;\,Z) = \sum_{n=0}^{\infty} \frac{(a)_n\,Z^n}{n!} = (1-Z)^{-a} \;. \qquad (1.5.6)$$

(v) Sonine or Generalised or Associated Laguerre polynomials

$$ L_n^{\alpha}(x) = \frac{(1+\alpha)_n}{n!}\;{}_1F_1\left[\begin{matrix} -n\;; \\ 1+\alpha\;; \end{matrix}\; x\right] ,$$

(vi) Simple Laguerre's polynomials

$$ L_n^{o}(x) = L_n(x) = {}_1F_1\left[\begin{matrix} -n\;; \\ 1\;; \end{matrix}\; x\right] .$$

(vii) Hermite polynomials

$$H_n(x) = (2x)^n \, {}_2F_0\left(-\frac{n}{2}, \frac{-n}{2}+\frac{1}{2}; -; -\frac{1}{x^2}\right),$$

$$H_{2n}(x) = \frac{(-1)^n (2n)!}{n!} \, {}_1F_1(-n; 1/2; x^2),$$

$$H_{2n+1}(x) = \frac{(-1)^n \, 2x \, (2n+1)!}{n!} \, {}_1F_1(-n; 3/2; x^2).$$

(viii) **Jacobi polynomials**

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_2F_1\left[\begin{matrix} -n, \; n+\alpha+\beta+1 \; ; \\ \alpha+1 \; ; \end{matrix} \; \frac{1-x}{2}\right]$$

$$= \frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n, \; -\beta-n \; ; \\ 1+\alpha \; ; \end{matrix} \; \frac{x-1}{x+1}\right]$$

$$= \frac{(-1)^n (1+\beta)_n}{n!} \, {}_2F_1\left[\begin{matrix} -n, \; 1+\alpha+\beta+n \; ; \\ 1+\beta \; ; \end{matrix} \; \frac{1+x}{2}\right]$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!\,(1+\alpha+\beta)_n}\left(\frac{x-1}{2}\right) {}_2F_1\left[\begin{matrix} -n, \; -\alpha-n \; ; \\ -\alpha-\beta-2n \; ; \end{matrix} \; \frac{2}{1-x}\right]$$

$$= \frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n, \; -\alpha-n \; ; \\ 1+\beta \; ; \end{matrix} \; \frac{x+1}{x-1}\right]$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!\,(1+\alpha+\beta)_n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n, -\beta-n \; ; \\ -\alpha-\beta-2n \; ; \end{matrix} \; \frac{2}{1+x}\right].$$

Note : $P_n^{(\alpha,\alpha)}(x)$ is called ultra spherical polynomials

and $P_n^{(o,o)}(x) = P_n(x)$

(ix) Associated Legendre's polynomials

$$P_n^m(x) = \frac{(m+n)!}{(n-m)!} \frac{(1-x^2)^{m/2}}{2^m\, m!} {}_2F_1\left(m-n,\ m+n+1;\ m+1;\ \frac{1-x}{2}\right).$$

Note $P_n^o(x) = P_n(x)$.

(x) Legendre's polynomials

$$P_n(x) = {}_2F_1\left(-n,\ n+1\ ;\ 1;\ \frac{1-x}{2}\right)$$

$$= \frac{(1/2)_n (2x)^n}{n!}\ {}_2F_1\left[\begin{matrix}-n/2,\ -\frac{n}{2}+1\ ; \\ 1/2-n\ ;\end{matrix}\ \frac{1}{x^2}\right]$$

$$= \left(\frac{x-1}{2}\right)^n {}_1F_1\left[\begin{matrix}-n,\ -n; \\ 1\ ;\end{matrix}\ \frac{x+1}{x-1}\right]$$

$$= x^n \cdot {}_2F_1\left[\begin{matrix}-n/2,\ -n/2+1/2\ ; \\ 1\ ;\end{matrix}\ \frac{x^2-1}{x^2}\right].$$

(xi) Tchebichef polynomials

I kind

$$T_n(x) = {}_2F_1\left(-n,\ n;\ 1/2\ ;\ \frac{1-x}{2}\right).$$

II kind

$$U_n(x) = \sqrt{(1-x^2)}\ n!\ {}_2F_1\left(-n+1,\ n+1;\ 3/2;\ \frac{1-x}{2}\right).$$

(xii) Gegenbauer polynomials

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}\ {}_2F_1\left(-n,\ n+2\lambda;\ \lambda + 1/2\ ;\ \frac{1-x}{2}\right).$$

(xiii) Incomplete gama function

$$\Gamma(\alpha, x) = \int_0^x e^{-t}\, t^{\alpha-1}\, dt = \alpha^{-1}\, x^{\alpha}\ {}_1F_1(\alpha;\ \alpha+1\ ;\ -x)\ ;$$

$Re(\alpha) > 0$.

(xvi) Error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt = \frac{2x}{\sqrt{\pi}}\ {}_1F_1\left(\frac{1}{2}\ ;\ \frac{3}{2};\ -x^2\right).$$

Note $\operatorname{erf}(\infty) = 1$.

(xv) Complete elliptic integral of I kind

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}\ ;\ \frac{1}{2}\ ;\ 1;\ k^2\right),$$

$0 < k < 1$.

(xvi) Complete elliptic integral of II kind

$$E(k) = \int_0^{\pi/2} \sqrt{(1-k^2 \sin^2 \phi)}\, d\phi$$

$$= \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1, k^2\right); \; 0 < k < 1 .$$

CHAPTER 2

FOUR DIMENSIONAL LIE ALGEBRA AND SPECIAL FUNCTIONS

2.1 Introduction

The theory of special functions has made use of techniques from the theory of analytic functions. There are properties of special functions which by their nature can be handled analytically by using integral representations, transformations, reductions, summations and expansions. However, it is seen that a great many of the properties which appear in the form of expansions, summations and addition theorems may be derived without any recourse to analytic methods. These methods are replaced by the powerful concepts of the Lie algebra or Lie group related to the special functions under consideration. A similar point of view was taken by Infeld and Hull [20] in their paper on factorization method. The operators which raise and lower indices in special functions were considered as generators of Lie algebra by Kaufman [26] for obtaining certain properties of special functions of mathematical physics from the viewpoint of Lie algebra. Weisner [50], [51], [52] obtained generating functions for special functions by using the concept of Lie groups.

The close relationship of Infeld and Hull's work [20] with Lie groups has also been noted by Miller [31]. In his Memoir he again classified the work of Infeld and Hull, into a classification of Lie algebras, all of them being special cases of a master group with four parameters. This 4-dimensional Lie algebra $\mathcal{G}(a,b)$ is introduced in section 2.2. In later sections, we construct realizations of $\mathcal{G}(a,b)$ by means of generalized Lie derivatives in two complex variables. Our work makes no attempt at an overall classification similar to Miller [31], [33] but rather takes any set of general differential operators and the corresponding recurrence relations as given. Machinery constructed in section 2.2 will then be applied to a special class of polynomials $S_n(x)$ introduced in section 2.3. From each set of recurrence relations the corresponding Lie algebra is generated and a multiplier representation of a three dimensional complex local Lie group T_3 is obtained. In the last section 2.5, we obtain generating functions for $S_n(x)$ by using the multiplier representation of T_3. It has been also shown that many known generating functions become particular cases of our main result.

2.2 The four dimensional complex Lie algebra

For any pair of complex numbers (a,b), define the four dimensional complex Lie algebra $\mathcal{G}(a,b)$ with basis J^+, J^-, J^3, $\mathcal{E}$ satisfying

$$[J^3, J^{\pm}] = \pm J^{\pm}, \; [J^+, J^-] = 2\,a^2 J^3 - b\,\xi,$$
$$[J^+, \xi] = [J^-, \xi] = [J^3, \xi] = \mathcal{O}, \tag{2.2.1}$$

where $\mathcal{O}$ is the additive identity element. Relations (2.2.1) define a Lie algebra (see Miller [31, p. 36]). For special values of the parameters a,b, $\mathcal{G}(a,b)$ coincides with one of the Lie algebras, $g\ell(2)$, $s\ell(2)$, $L(\mathcal{G}(0,1))$, τ_3 introduced in Chapter 1. The following result (Miller [31, p. 37]) :

<u>Lemma I</u>

$$\mathcal{G}(a,b) = \begin{cases} \mathcal{G}(1,o) & \text{if } a \neq o \\ \mathcal{G}(o,1) & \text{if } a = o,\ b \neq o \\ \mathcal{G}(o,o) & \text{if } a = b = o \end{cases}$$

shows that there are only three distinct Lie algebras of the form $\mathcal{G}(a,b)$ up to isomorphism.

Let ρ be a representation of $\mathcal{G}(a,b)$ on the complex vector space V and set the differential operators

$$J^+ = \rho(J^+), \; J^- = \rho(J^-), \; J^3 = \rho(J^3), \; E' = \rho(\xi).$$

These linear operators obey the commutation relations

$$[J^+, J^-] = 2a^2 J^3 - bE', \; [J^3, J^{\pm}] = \pm J^{\pm},$$
$$[J^+, E'] = [J^-, E'] = [J^3, E'] = 0, \tag{2.2.2}$$

where $[A,B] = AB - BA$.

To obtain realizations of the irreducible representations ρ of $\mathcal{G}(a,b)$ as listed in section 2.6 of Miller [31, pp. 38-44] such that V becomes a vector space of analytic functions and the operators $\rho(\alpha)$, $\alpha \in \mathcal{G}(a,b)$ form a Lie algebra of analytic differential operators acting on V, we will determine $\rho(x)$ in general and a number of solutions of equations (2.2.2) in particular.

In addition to linear operators such as [31, p. 45]

$$J^3 = \frac{\partial}{\partial y}, \quad J^{\pm} = e^{\pm y}\left(\pm\frac{\partial}{\partial x} - k(x)\frac{\partial}{\partial y} + j(x)\right), \quad E' = \mu, \tag{2.2.3}$$

where μ is a complex constant and k, j are functions of x to be determined, an important role is played by linear differential operators of the following nature in the theory of special functions

$$\begin{aligned} J^3 &= A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C, \\ J^- &= E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y} + G, \\ J^+ &= R\frac{\partial}{\partial x} + S\frac{\partial}{\partial y} + T, \\ E' &= \mu I, \end{aligned} \tag{2.2.4}$$

where A,B,C ; E,F,G ; R,S,T are functions of x and y and μ is a complex constant. For special choices of A,B,C ; E,F,G ; R,S,T, (2.2.4) coincides with (2.2.3). Because of the important role which operators of (2.2.4) for general A,B,C ; E,F,G ; R,S,T play in special functions, the significance of (2.2.4) becomes more understandable. In (2.2.3), restrictive assumptions are made to the form of the operators $\rho(\alpha)$ and are probably of limited interest.

The operators (2.2.4) satisfy all of the commutation relations (2.2.2) provided that

I

$$A\partial R/\partial x + B\partial R/\partial y - R\partial A/\partial x - S\partial A/\partial y = R$$

$$A\partial S/\partial x + B\partial S/\partial y - R\partial B/\partial x - S\partial B/\partial y = S$$

$$A\partial T/\partial x + B\partial T/\partial y - R\partial C/\partial x - S\partial C/\partial y = T \,.$$

II

$$A\partial E/\partial x + B\partial E/\partial y - E\partial A/\partial x - F\partial A/\partial y = -E$$

$$A\partial F/\partial x + B\partial F/\partial y - E\partial B/\partial x - F\partial B/\partial y = -F$$

$$A\partial G/\partial x + B\partial G/\partial y - E\partial C/\partial x - F\partial C/\partial y = -G \,.$$

III

$$R\partial E/\partial x + S\partial E/\partial y - E\partial R/\partial x - F\partial R/\partial y = 2a^2A$$

$$R\partial F/\partial x + S\partial F/\partial y - E\partial S/\partial x - F\partial S/\partial y = 2a^2B \,.$$

$$R\partial G/\partial x + S\partial G/\partial y - E\partial I/\partial x - F\partial I/\partial y = 2a^2 C - b\mu . \quad (2.2.5)$$

Suppose we are able to realize an irreducible representation ρ of $\mathcal{G}(a,b)$ in such a way that the operators satisfy (2.2.2) and the basis space V is a space of analytic functions of x and y. Now two cases arise

(i) The basis functions $f_m(x,y)$, $m \in S$ where S is the spectrum, satisfy the equations

$$J^3 f_m(x,y) = (A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C)\, f_m(x,y) = m\, f_m(x,y). \quad (2.2.6)$$

Then $f_m(x,y) = g_m(x)\, e^{my}$ for all $m \in S$, where $g_m(x)$ is an analytic function of x. Since ρ is irreducible and Casimir operator is given by [31]

$$C_{a,b} = J^+J^- + a^2 J^3J^3 - a^2J^3 - bJ^3E', \quad (2.2.7)$$

we have

$$C_{a,b}\,[\,f_m(x,y)] = \lambda\, f_m(x,y),\ m \in S , \quad (2.2.8)$$

where the constant λ is uniquely determined by ρ . The function $g_m(x)$ is a special function and the action of ρ would give us generating functions and addition theorems for this special function.

(ii) The basis functions $f_m(x,y)$ satisfy the equations

(2.2.6) but $f_m(x,y) = y^m g_m(x)$ for all $m \in S$ and $f_m(x,y)$ is a solution of a second order differential equation to be determined. The Casimir operator (2.2.7) enables us to write

$$C_{a,b}\,[f_m(x,y)] = \lambda\, f_m(x,y),\ m \in S\ , \tag{2.2.9}$$

where the constant λ is uniquely determined by ρ.

To determine the possible functions $g_m(x)$, we find $C_{a,b}$ and evaluate (2.2.8) or (2.2.9) for our given operators. Clearly, $C_{a,b}$ depends on a and b and thus the following three different cases of $\mathcal{G}(a,b)$ are obtained.

$\underline{\mathcal{G}(1,o)}$. For a = 1, b = o, equations (2.2.2) reduce to

$$[J^+,J^-] = 2J^3,\ [J^3,\ J^{\pm}] = \pm\, J^{\pm}. \tag{2.2.10}$$

We now consider a few special cases.

<u>Case 1</u>. On choosing $A = o$, $B = y$, $C = \frac{\alpha+1}{2}$; $E = \frac{x}{y}$, $F = -1$, $G = o$; $R = xy$, $S = y^2$, $T = (1 + \alpha - x)\,y$, we see that equations (2.2.5) and (2.2.10) are satisfied. Also equations (2.2.10) are identical with the commutation relations for the generators of $L[SL(2)] = s\ell(2)$. With this choice of A,B,C ; E,F,G ; R,S,T, equations (2.2.4) take the form

$$J^3 = y\frac{\partial}{\partial y} + \frac{\alpha+1}{2},$$

$$J^- = \frac{x}{y}\frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$

$$J^+ = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (1+\alpha-x)y. \tag{2.2.11}$$

Now we introduce the first order linearly independent differential operators J^3, J^+ and J^-, each of the form

$$A_1(x,y)\frac{\partial}{\partial x} + A_2(x,y)\frac{\partial}{\partial y} + A_3(x,y) \tag{2.2.12}$$

such that

$$J^3[y^n g_n(x)] = a_n\, y^n\, g_n(x),$$

$$J^-[y^n g_n(x)] = b_n\, y^{n-1}\, g_{n-1}(x),$$

$$J^+[y^n g_n(x)] = c_n\, y^{n+1}\, g_{n+1}(x), \tag{2.2.13}$$

where a_n, b_n and c_n are expressions in n which are independent of x and y. Each $A_i(x,y)$, $i = 1, 2, 3$, on the other hand, is an expression in x and y which is independent of n.

On using (2.2.11) for $J^{\pm}$, J^3 in (2.2.13), we get the following recurrence relations

$$\frac{d}{dx} g_n(x) = \frac{1}{x}[n\, g_n(x) - n\, g_{n-1}(x)],$$

$$\frac{d}{dx} g_n(x) = \frac{1}{x} [(x-n-1)\, g_n(x) + (n-1)\, g_{n+1}(x)] . \qquad (2.2.14)$$

We observe that these recurrence relations of $g_n(x)$ coincide with the known recurrence relations of the associated Laguerre polynomials $L_n^{\alpha}(x)$.

Also the Casimir operator

$$\mathcal{C}_{1,0} = J^+J^- + J^3 J^3 - J^3$$

$$= x^2 \frac{\partial^2}{\partial x^2} + x\,(1+\alpha-x) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + \frac{\alpha^2-1}{4} \qquad (2.2.15)$$

commutes with J^3, J^+ and J^- because of the relations (2.2.10) and enables us to rewrite

$$\mathcal{C}_{1,0} [f_n(x,y)] = \left(\frac{\alpha^2-1}{4} \right) f_n(x,y) ,$$

which is of the form of (2.2.9). Here we have used

$$f_n(x,y) = y^n \mathcal{U}(x) ,$$

where $\mathcal{U}(x) = L_n^{\alpha}(x)$ and the differential equation ([38], p. 204) for $L_n^{\alpha}(x)$ is of the form

$$[x \frac{\partial^2}{\partial x^2} + (1+\alpha-x) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}]\, f_n(x,y) = 0 . \qquad (2.2.16)$$

Case 2. On choosing $A = o$, $B = y$, $C = m + \frac{\gamma+1}{2}$, $E = xy$, $F = y^2$, $G = (\gamma+1+m-x)y$, $R = x/y$, $S = -1$, $T = -m/y$, we observe that equations (2.2.5) and (2.2.10) are satisfied. Thus we conclude that these J operators form a Lie algebra isomorphic to $s\ell(2)$, Lie algebra of SL(2).

Now following the method of Case 1, we find that our special functions $f_n(x,y)$ are associated Laguerre functions $y^n L^{\gamma}_{m+n}(x)$.

Such J operators were first considered by Manocha and Jain [23] to obtain certain generating functions for associated Laguerre functions $L^{\gamma}_{m+n}(x)$.

Case 3. Another example of J operators forming a Lie algebra isomorphic to $s\ell(2)$ is obtained by choosing

$$A = o, \qquad B = y, \qquad C = m + \gamma/2\,,$$

$$E = x/y, \qquad F = -m/y, \qquad G = o\,,$$

$$R = xy(1-x), \quad S = y^2, \qquad T = (\gamma+m-\beta x)y,$$

which was considered by S. Jain [22] for getting generating functions for hypergeometric functions ${}_2F_1(-m-n,\ \beta,\ \gamma;\ x)$.

Case 4. On choosing

$$A = o, \qquad B = y, \qquad C = 1/2\,,$$

$$E = \frac{x^2-1}{y}, \qquad F = -x, \qquad G = o,$$

$$R = (x^2-1)y, \quad S = y^2 x, \qquad T = xy,$$

we see that equations (2.2.5) and (2.2.10) are satisfied. Thus our J operators form a Lie algebra isomorphic to sl (2). Operators $J^{\pm}$, J^3 yield recurrence relations for the functions which are identified as Legendre polynomials $P_n(x)$. The Casimir operator

$$C_{1,o}\,[f_m(x,y)] = 1/4\, f_m(x,y),\ m \in S$$

satisfies (2.2.9).

$\underline{\mathcal{G}(o,1)}$. If a = o, b = 1, equations (2.2.2) yield

$$[J^+, J^-] = -bE', \ [J^3, J^{\pm}] = \pm J^{\pm},$$

$$[J^{\pm}, E'] = [J^3, E'] = o\,. \tag{2.2.17}$$

These equations are identical with the commutation relations for the generators of $L[G(o,1)]$.

On choosing

$$A = o, \ \ B = y, \ \ C = o; \ E = 1/y, \ \ F = G = o\ ; \ R = -xy,$$

$$S = y^2, \ T = (x-\alpha)y, \ E' = \mu = 1,$$

we see that equations (2.2.5) are satisfied. Recurrence relations of $g_n(x)$ coincide with the known recurrence relations of the associated laguerre polynomials $L_n^{a-n}(x)$.

The operators with these values of A,B,C ; E,F,G ; R,S,T were considered by Manocha [23].

$\mathcal{G}(o,o)$. If a = b = o, equations (2.2.2) yield

$$[J^+, J^-] = o, \quad [J^3, J^{\pm}] = \pm J^{\pm}. \tag{2.2.18}$$

These equations are identical with commutation relations for the generators of $L[T_3] = \tau_3$.

We consider the following special cases of $\mathcal{G}(o,o)$.

Case 1. If the differential operators are such that

$$A = o, \; B = 1, \; C = o; \; E = -e^{-y}, \; F = -e^{-y}/x \;, \; G = o;$$

$$R = e^{y}, \; S = -e^{y}/x, \; T = o,$$

then these operators are type C'' operators classified in section 2.7 of Miller [31].

2.3 A special class of polynomials $S_n(x)$

The machinery constructed in section 2.2 will here be applied to the study of a special class of polynomials $S_n(x)$. As was demonstrated in section 2.2. These polynomials are shown to be related to the representation theory of $\mathcal{G}(0,0)$. We will now try to find new operators which satisfy (2.2.2) and (2.2.3) for $a = b = 0$. In particular, we look for functions $f_m(x,y) = e^{my} Z_m(x)$, such that

$$J^3 f_m = m f_m \ , \ J^+ f_m = \nu f_{m+1} , \ J^- f_m = \nu f_{m-1} \ ,$$

$$C_{0,0} f_m = J^+ J^- f_m = \nu^2 f_m , \ \nu \neq 0 \tag{2.3.1}$$

for all $m \in S = \{m_0 + k \ ; \ k \text{ an integer}\}$. The primary tools, notations and definitions needed to deduce our results are first introduced.

W. Schultz-Piszachuch [40] tried to construct a formal theory of a certain family of isotropic turbulence field and introduced a special class of polynomials $S_n(x)$ which occur in the following representation of the energy spectrum functions [cf [40], p. 312, equation (70)]

$$E_n(k,t) = \bar{u}^2 a_n^5 k^4 \exp(-a_n k) \ S_{n-3}(a\,k), \ n \geqslant 3 \ .$$

Problems connected with a series representation for the polynomial $S_n(x)$ were taken up by U. Werner and W. Pietzsch [[53], p. 167, equation (9)]

$$S_n(x) = \sum_{k=0}^{n} \frac{1}{2^{n-k}} \ \frac{(2n-k)!\ x^k}{(n-k)!\ k!} \ . \tag{2.3.2}$$

They went to great length to use this representation to obtain several important properties of $S_n(x)$, including, for example, recursion formula, a differential equation and an integral representation for $S_n(x)$.

Srivastava [42] pointed out the fact that the polynomials $S_n(x)$ are certain special cases of the classical Laguerre polynomials $L_n^{\alpha}(x)$ defined by

$$L_n^{\alpha}(x) = \sum_{n=0}^{\infty} \binom{n+\alpha}{n-\alpha} \frac{(-x)^k}{k!} \cdot \tag{2.3.3}$$

Indeed the polynomials $S_n(x)$ are also contained in the Srivastava-Singhal polynomials ([45, p. 75, equation (3))

$$G_n^{(\alpha)}(x,r,p,s) = \frac{1}{n!} x^{-\alpha-sn} \exp(px^r)(x^{s+1}D_x)^n[x^{\alpha}\exp(-px^r)], \tag{2.3.4}$$

where the parameters α, r, p and s are, in general, unrestricted. Furthermore, the polynomials $S_n(x)$ are related rather closely to the familiar Bessel polynomials $y_n(x)$ defined by ([27], p. 101, equation (3)).

$$y_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} k! \left(\frac{x}{2}\right)^k, \tag{2.3.5}$$

a historical sketch of whose occurrences in the mathematical literature since 1873 is included in the excellent monograph on the subject by E. Grosswald [16].

Srivastava [42] applied aforementioned connections in simple derivations of various important and useful properties of polynomials $S_n(x)$.

Since $$(N-k)! = \frac{(-1)^k N!}{(-N)_k}, \quad 0 \leq k \leq N, \tag{2.3.6}$$

where, for convenience,

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma\lambda} = \begin{cases} 1, \text{ if } k = 0 \\ \lambda(\lambda+1) \ldots (\lambda+k-1), \text{ for all } k \in \{1,2,3\ldots\}, \end{cases} \tag{2.3.7}$$

the series representation (2.1.2) may be rewritten in the form

$$S_n(x) = \frac{(2n)!}{2^n\, n!} \sum_{k=0}^{n} \frac{(-n)_k}{(-2n)_k} \frac{(2x)^k}{k!}, \tag{2.3.8}$$

it being understood, as also in (2.3.2), that $S_0(x) = 1$. A comparison between (2.3.3) and (2.3.8) readily yields the relationship

$$S_n(x) = n!\,(-2)^{-n(-2n-1)}\, L_n(2x) \;, \tag{2.3.9}$$

which exhibits the interesting fact that the polynomials $S_n(x)$ are constant multiple of the classical Laguerre polynomials $L_n^{\alpha}(x)$ with, of course, $\alpha = -2n-1$ and x replaced by $2x$.

On comparing the definitions $S_n(x) = (-1)^n x^{2n+1} e^x (x^{-1} D_x)^n \{x^{-1} e^{-x}\}$ and (2.3.4) we obtain the relationship

$$S_n(x) = (-1)^n n! \, G_n^{(-1)} (x, 1, 1, -2) \qquad (2.3.10)$$

between the polynomials $S_n(x)$ and $G_n^{(\alpha)} (x,r,p,s)$.

Next, comparing the series representation (2.3.2) and (2.3.5), after having reversed the order of terms in one of them, we obtain

$$S_n(x) = x^n y_n (x^{-1}), \qquad (2.3.11)$$

which relates $S_n(x)$ with the familiar Bessel polynomials $y_n(x)$.

We first note that using one of the other of the relationships (2.3.9), (2.3.10) and (2.3.11), we can deduce, among several other results of interest, many basic properties of the polynomials $S_n(x)$ as immediate consequences of the corresponding known properties of the classical Laguerre polynomials, the Srivastava Singhal polynomials, or the Bessel polynomials (cf. Rainville [38], Chapter 12, Srivastava and Singhal [44], Krall and Frink [27], see also Szego [45] and Grosswald [16]). Now using the fact

$$xD_x [S_n(x)] = (x+2n+1) S_n(x) - S_{n+1}(x) , \qquad (2.3.12)$$

which follows at once upon suitably specializing the Srivastava-Singhal formula ([45], p. 80, equation (4.3)), we get the following recurrence relations

$$D_x[S_n(x)] = S_n(x) - x\,S_{n-1}(x) \quad , \tag{2.3.13}$$

$$S_n(x) = (2n-1)\,S_{n-1}(x) + x^2 S_{n-2}(x)\,, \tag{2.3.14}$$

$$D_x[S_n(x)] + x\,D_x[S_{n-1}(x)] = (2n-1)\,S_{n-1}(x), \tag{2.3.15}$$

$$(x+n)\,D_x\{S_n(x)\} + x^2\,D_x\{S_{n-1}(x)\} = n\,S_n(x) + (n-1)\,x\,S_{n-1}(x)\,, \tag{2.3.16}$$

$$2xD_x\{S_n(x)\} = (2x+2n+1)S_n(x) - x^2 S_{n-1}(x) - S_{n+1}(x)\,, \tag{2.3.17}$$

all of which are well known for the Bessel polynomials (cf. [27], p. 105, equation (21) see also ([16], p. 18-19).

The differential equation for $S_n(x)$ is

$$x\frac{d^2X}{dx^2} - 2(x+n)\frac{dX}{dx} + 2nX = 0, \quad X = S_n(x)\,, \tag{2.3.18}$$

which is the well known Laguerre equation ([38], p. 204, Section 116, equation (1)) with $\alpha = -2n-1$ and x replaced by 2x.

The generating functions

$$\sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} = (1-2t)^{-1/2} \exp\{x(1-(1-2t)^{1/2})\}, \quad (2.3.19)$$

$$\sum_{n=0}^{\infty} S_{m+n}(x) \frac{t^n}{n!} = (1-2t)^{-m-1/2} \exp(x[1-(1-2t)^{1/2}]) \, S_m(x(1-2t)^{1/2}) \quad (2.3.20)$$

follow directly from the known result ([44], p. 78, equation (3.2), p. 79, equation (3.6)) by appealing to relationship (2.3.10).

2.4 Multiplier representation of T_3

On choosing

$$A = o, \quad B = 1, \quad C = o,$$

$$E = -1/x\, e^{-y}, \quad F = o, \quad G = 1/x\, e^{-y},$$

$$R = -xe^{y}, \quad S = 2e^{y}, \quad T = (x+1)e^{y},$$

equations (2.2.5) are satisfied for $a = b = o$.

Hence our operators $J^{\pm}$, J^3 are given by

$$J^3 = \partial/\partial y,$$

$$J^+ = -xe^{y}\, \partial/\partial x + 2e^{y}\, \partial/\partial y + (x+1)\, e^{y},$$

$$J^- = -e^{-y}/x\, \partial/\partial x + e^{-y}/x. \quad (2.4.1)$$

Commutation relations of these operators are identical to equation (2.2.18).

We seek linearly independent differential operators J^3, J^+ and J^- each of the form.

$$A_1(x,y)\frac{\partial}{\partial x} + A_2(x,y)\frac{\partial}{\partial y} + A_3(x,y)$$

such that

$$J^3[y^n Z_n(x)] = a_n\, y^n Z_n(x)\,,$$

$$J^-[y^n Z_n(x)] = b_n\, y^{n-1} Z_{n-1}(x)\,, \tag{2.4.2}$$

and

$$J^+[y^n Z_n(x)] = c_n\, y^{n+1} Z_{n+1}(x)\;,$$

where a_n, b_n and c_n are expressions in n which are independent of x and y and each $A_i(x,y)$, $i = 1, 2, 3$ is an expression in x and y which is independent of n.

This makes it necessary to use the differential operators J^+, J^3 and J^- and it follows that the recurrence relations are identifiable as (2.3.12) and (2.3.13). Thus the functions $Z_n(x)$ turn out to be $S_n(x)$ introduced in section 2.2.

The comparison of (2.2.1) and (2.2.18) in conjugation with a Theorem of Miller (cf. [31], p. 18) suggests that operators J^3, J^-, J^+ in (2.4.1) act as generalized Lie derivatives for the multiplier representation $T : T_3 \longrightarrow F$, where F is the vector space of all analytic functions. Now we proceed to compute the multiplier representation of T_3. The actions of 1-parameter group $\exp \tau j^3$, $\exp c j^-$ and $\exp b j^+$ are obtained by integrating the following differential equations (cf. [31], p. 18).

$$\frac{dx(\tau)}{d\tau} = 0, \quad \frac{dy(\tau)}{d\tau} = 1, \quad \frac{d\nu(\tau)}{d\tau} = 0 \;.$$

$$\frac{dx(c)}{dc} = -e^{y(c)}/x(c), \quad \frac{dy(c)}{dc} = 0\;, \quad \frac{d\nu(c)}{dc} = \nu(c)e^{-y(c)}/x(c),$$

$$\frac{dx(b)}{db} = -x(b)\; e^{y(b)}, \quad \frac{dy(b)}{db} = 2e^{y(b)}, \quad \frac{d\nu(b)}{db} = [x(b)+1]e^{yb}\nu(b)$$

subject to the conditions $x(o) = x^o$, $y(o) = y^o$, $\nu(o) = 1$, where ν is multiplier of the representation.

Hence, the values of the multiplier representations of $\exp \tau j^3$, $\exp c j^-$ and $\exp b j^+$ are respectively given by

$$[T(\exp \tau j^3)f]\;(x^o,t^o) = f(x^o,\; e^{\tau}t^o)\,,$$

$$[T(\exp c j^-)f]\;(x^o,t^o) = \exp\;(x^o - (x^{o2} - 2c/t^o)^{1/2})$$

$$f(\sqrt{x^{o2} - 2c/t^o},\; t^o),$$

$$[T(\exp bj^{+})f]\,(x^{0},t^{0}) = \exp(-x^{0}t^{0\,1/2}(-2b+1/t^{0})^{1/2}+$$

$$+x^{0}-\log t^{0\,1/2})\,(-2b+1/t^{0})^{-1/2}f(x^{0}t^{0\,1/2}(-2b+1/t^{0})^{1/2},$$

$$(-2b+1/t^{0})^{-1}),$$

where $t^{0} = e^{y^{0}}$

If $g \in T_3$ is given by

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-z} & 0 & c \\ 0 & 0 & e^{z} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b, c, \tau \in \mathbb{C},$$

we find that

$$g = \exp(bj^{+})\exp(cj^{-})\exp(\tau j^{3}).$$

So, for $f \in cl$ where cl is the space of all analytic functions in some neighborhood of the point (x^{0}, y^{0}) and g in a sufficiently small neighborhood of the identity, we have

$$[T(g)f]\,(x,t)$$

$$= [T(\exp bj^{+})\,T(\exp cj^{-})\,T(\exp\tau j^{3})f](x,t)$$

$$= (1-2bt)^{-1/2}\exp[x-(-2b+1/t)^{1/2}(-2c+x^{2}t)^{1/2}]$$

$$\cdot f((-2b+1/t)^{1/2}(-2c+x^{2}t)^{1/2},\; e^{\tau}(-2b+1/t)^{-1}). \qquad (2.4.3)$$

2.5 Generating Functions of $S_n(x)$

Now we apply the representation theory of the Lie group T_3 developed in § 2.4 to obtain generating functions of $S_n(x)$.

The machinery constructed in Miller ([31], Chapters 1 and 2) will be applied to find realization of the representation $Q(w, m_0)$ of $\mathcal{T}_3$ where w, m_0 are complex constants such that $w \neq 0$ and $0 \leqslant \mathrm{Re}\, m_0 < 1$. The spectrum S of this representation is the set $\{m_0 + k : k \text{ an integer}\}$. In particular, we look for the functions $f_m(x,y) = Z_m(x)\, e^{my}$, such that equation (2.3.1) are satisfied for $w = 1$ and for all $m \in S$, where $J^{\pm}$, J^3 are given by (2.4.1).

According to Miller ([31], p. 27, § 2.2), our realization of the representation $Q(1, m_0)$ of $\mathcal{T}_3$ on the space generated by the function $f_m(x,y)$, $m \in S$ can be extended to a local representation T_3 where the group action is given by (2.4.3). The matrix elements of this local representation with respect to the basis f_m are uniquely determined by $Q(1, m_0)$ and we obtain the relations

$$[T(g)f_{m_0+k}](x,t) = \sum_{l=-\infty}^{\infty} A_{lk}(g)\, f_{m_0+l}(x,t), k = 0, \pm 1, \pm 2 \tag{2.5.1}$$

$$t^{-1/2}(t^{-1}-2b)^{-1/2-m}\exp\,[m\tau + x - (t^{-1}-2b)^{1/2}(x^2t-2c)^{1/2}]$$

$$\cdot\, S_m\,((t^{-1}-2b)^{1/2}(x^2t-2c)^{1/2})$$

$$=\sum_{l=-\infty}^{\infty} A_{l,m-m_0}(g)\; S_{m_0+l}(x)\; t^{m_0\,l}, \tag{2.5.2}$$

where the matrix elements $A_{lk}(g)$ are given by ([31], p. 56, equation (3.12)).

$$A_{lk}(g) = \frac{e^{(m_0+k)\tau}\, c^{(k-l+|k-l|)/2}\, b^{(l-k+|k-l|)/2}}{|k-l|!}$$

$$\cdot\, {}_0F_1\,(|k-l|+1\; ;\; bc)\;, \tag{2.5.3}$$

valid for all integral values of l, k. Since $S_m(x)$, $m\in\mathbb{C}$ is analytic in x for all nonzero values of x, the infinite series (2.5.2) converges absolutely for $|2bt\;|<1$, $|2c/x^2t|<1$. Thus our main generating function becomes

$$(-2bt+1)^{-m-1/2}\exp\,[x-(-2bt+1)^{1/2}(-2c/t+x^2)^{1/2}]$$

$$\cdot\, S_m\,((-2bt+1)^{1/2}\,(-2c/t+x^2)^{1/2})$$

$$=\sum_{n=-\infty}^{\infty} c^{(-n+|n|)/2}\, b^{(n+|n|)/2}\; {}_0F_1\,(|n|+1\; ;\; bc)\; S_{m+n}(x)t^n. \tag{2.5.4}$$

Several results of special functions are particular cases of the formula (2.5.4). If c = o and b = 1, equation (2.5.4) becomes

$$(1-2t)^{-m-1/2} \exp\,[x(1-(1-2t)^{1/2})]\; S_m\,[x(1-2t)^{1/2}]$$

$$= \sum_{n=0}^{\infty} S_{m+n}(x)\,\frac{t^n}{n!}\,, \qquad (2.5.5)$$

where m = o, 1, 2 ... (2.5.5) is a known generating function [See Srivastava ([42], p. 256, equation (36)] and also follows from the known results of Srivastava and Singhal [44], p. 78, equation (3.2), p. 79, equation (3.6)].

A special case of (2.5.5),

$$\sum_{n=0}^{\infty} S_n(x)\,\frac{t^n}{n!} = (1-2t)^{-1/2} \exp\,[x\,(1-(1-2t)^{1/2})]$$

can also be derived by suitably specializing a result due to Carlitz ([8], p. 826, equation (8)).

If b = o and t = 1 in (2.5.4), we obtain

$$\exp\,[x-(x^2-2c)^{1/2}]\; S_m\,[(x^2-2c)^{1/2}]$$

$$= \sum_{n=0}^{\infty} \frac{c^n}{n!}\, S_{m-n}(x)\,. \qquad (2.5.6)$$

When c = o, (2.5.4) yields a more familiar result

{Mc Bride [30], p. 50, equation (12)} for the simple Bessel polynomials $y_n(x)$

$$\sum_{n=0}^{\infty} y_{m+n}(x)\,\frac{t^n}{n!}$$

$$= (1-2\,xt)^{-(m+1/2)} \exp\left[\frac{1-(1-2xt)^{1/2}}{x}\right] y_m\left(x(1-2xt)^{1/2}\right).$$

(2.5.7)

If $bc \neq o$, we can introduce the coordinates r, γ defined by $r = (ibc)^{1/2}$ and $\gamma = (b/ic)^{1/2}$, such that $b = r\gamma/2$, $c = -r/2\gamma$. In that case equation (2.5.4) yields the generating function

$$(1-r\gamma t)^{-m-1/2} \exp\left[x-(1-r\gamma t)^{1/2}(x^2+r/\gamma t)^{1/2}\right]$$

$$S_m\left[(1-r\gamma t)^{1/2}(x^2-r/\gamma t)^{1/2}\right] = \sum_{n=-\infty}^{\infty} (-\gamma)^n J_n(r) S_{m+n}(x) t^n,$$

(2.5.8)

where $J_n(r)$ is a Bessel function of integral order.

For $m = o$, (2.5.4) gives

$$(1-2bt)^{-1/2} \exp\left[x-(1-2bt)^{1/2}(x^2-2c/t)^{1/2}\right]$$

$$= \sum_{n=-\infty}^{\infty} c^{(-n+|n|)/2}\, b^{(n+|n|)/2}\; {}_0F_1(|n|+1;\, bc)\; S_n(x)\; t^n\ .$$

(2.5.9)

CHAPTER 3

Q(0,1) AND GENERALIZED HERMITE POLYNOMIALS

3.1 Introduction

The natural generalizations of Hermite polynomials $H_n(x)$ [38] are suggested by a number of results in the theory of special functions. A generalization to $H_n^{\alpha}(x)$ for real x was given in 1976 by Pearce and Potts [35] in the form

$$H_{2n}^{\alpha}(x) = 2^{2n} \Psi(-n, \alpha+1; x^2) = (-1)^n 2^{2n} n! L_n^{\alpha}(x^2) \tag{3.1.1}$$

$$H_{2n+1}^{(\alpha)}(x) = 2^{2n+1} x \Psi(-n, \alpha+2; x^2) = (-1)^n 2^{2n+1} n! x L_n^{1+\alpha}(x^2), \tag{3.1.2}$$

where Ψ is the confluent hypergeometric function [38] and $L_n^{(\alpha)}(x)$ represents generalized Laguerre polynomials. The generalized Hermite polynomials $[H_n^{(\mu)}]_{n=0}^{\infty}$ given by

$$H_{2n}^{(\mu)}(x) = \sum_{k=0}^{n} \binom{n+\mu-\frac{1}{2}}{n-k} \frac{(-1)^{n-k} 2^{2n} n!}{k!} x^{2k} \tag{3.1.3}$$

$$H_{2n+1}^{(\mu)}(x) = \sum_{k=0}^{n} \binom{n+\mu+\frac{1}{2}}{n-k} \frac{(-1)^{n-k} 2^{2n+1} n! x^{2k+1}}{k!}, \tag{3.1.4}$$

(these relations can be found in Chihara ([10] pages 156–158)) have not been studied so extensively as their simpler counter parts $[H_n^{(o)}]_{n=o}^{\infty}$ or the other classical orthogonal polynomials system. This is largely due to their failure to satisfy a differential equation of the form $Ly = \lambda_n y$ where n appears only in the eigenvalues λ_n. Bochner [3] has shown that the classical polynomials $f_n(x)$ are completely characterised by satisfying the differential equation of the form

$$A(x)\, y'' + B(x)\, y' + \lambda_n\, y = o\,, \tag{3.1.5}$$

where A and B are independent of n and λ_n is independent of X. Pearce and Potts [35] have shown that $H_n^{\alpha}(x)$ do not satisfy such equation and therefore these polynomials should not be taken as being in any real sense equivalent to a classical system. Thus $H_n^{\alpha}(\alpha)$ contrasts sharply with the behaviour of classical orthogonal polynomials.

An attractive feature of the particular system of (3.1.1) and (3.1.2) is that the corresponding Lie group for the classical Hermite polynomials is a natural quotient subgroup of G(o,1). Thus various aspects of group structure for these functions are of interest, analogous to, say, the traditional role of Bessel functions as corresponding to radical part of the relevant motion group. The Bessel functions appear in two distinct ways : as matrix elements of local irreducible representation of $G(o,o)$ and as basis functions for irreducible representations of $L[G(o,o)]$ [see Miller [31], Chapters 2 and 3]. Since $L[G(o,o)]$ is a contraction of $L[G(0,1)]$, their relationship will turn out to be of fundamental importance in special function theory.

A number of facts are known for generalized Hermite polynomials. First they satisfy a pair of differential equations,

$$xy'' + 2(\mu - x^2)\, y' + (2nx - Q_n x^{-1})\, y = o \,, \qquad (3.1.6)$$

where $Q_{2n} = o$ and $Q_{2n+1} = 2\mu$. Only when $\mu = o$ do these coincide.

Secondly, the polynomials $[H_n^{(\mu)}(x)]$ are orthogonal with respect to the weight function $w(x) = \exp(-x^2)|x|^{\mu}$ and the orthogonality relation is given by

$$\int_{-\infty}^{\infty} \exp(-x^2)|x|^{\mu} H_m^{(\mu)}(x)\, H_n^{(\mu)}(x)\, dx = o, \; m \neq n \quad (3.1.7)$$

where μ is the parameter.

To suit our purpose, we shall use the following form of generalized Hermite polynomials

$$H_{2n}^{\mu}(x) = (-1)^n \left(\frac{\mu+1}{2}\right)_n \psi\left(-n, \frac{\mu+1}{2}, x^2\right)$$

$$= (-1)^n \, n! \, L_n^{\left(\frac{\mu-1}{2}\right)}(x^2) , \tag{3.1.8}$$

$$H_{2n+1}^{\mu}(x) = (-1)^n \left(\frac{\mu+3}{2}\right)_n x \, \psi\left(-n, \frac{\mu+3}{2}, x^2\right)$$

$$= (-1)^n \, n! \, x \, L_n^{\frac{\mu+1}{2}}(x^2), \tag{3.1.9}$$

in place of (3.1.1) and (3.1.2). Polynomials (3.1.8) and (3.1.9) first appeared in Szego's book ([46], p. 377). Later on Chihara studied their properties in his thesis submitted to Purdue University in 1955. For various interesting properties and generating functions of these polynomials, we refer More [34], Dutta, Chatterjea and More [11], Rai and Singh [37] and Srivastava [41].

Setting $\mu = 0$ in (3.1.8) and (3.1.9), we find that

$$\begin{aligned} H_{2n}^{0}(x) &= 2^{-2n} H_{2n}(x) , \\ H_{2n+1}^{0}(x) &= 2^{-(2n+1)} H_{2n+1}(x) , \end{aligned} \tag{3.1.10}$$

where $H_{2n}(x)$ and $H_{2n+1}(x)$ are the (ordinary) even and odd Hermite polynomials.

From (3.1.8) and (3.1.9), it readily follows that

$$H_0^{\mu}(x) = 1,\ H_1^{\mu}(x) = x,\ H_2^{\mu}(x) = x^2 - \frac{\mu+1}{2},$$

$$H_3^{\mu}(x) = x\left(x^2 - \frac{\mu}{2} - \frac{3}{2}\right),\ H_4^{\mu}(x) = x^4 - (\mu+3)\,x^2$$

$$+\left(\frac{\mu+1}{2}\right)\left(\frac{\mu+3}{2}\right).$$

Also

$$H_{2n}^{\mu}(o) = \left(\frac{\mu+1}{2}\right)_n (-1)^n;\ H_{2n+1}^{\mu}(o) = o\,.$$

Carlitz [8] has shown the following operational representation for Laguerre polynomials

$$L_n^{\alpha}(x) = \frac{1}{n!}\prod_{j=1}^{n}(xD - x + \alpha + j)J\ , \qquad (3.1.11)$$

from which we can easily obtain

$$L_n^{\alpha}(x) = \frac{1}{n!}\, e^x \prod_{j=1}^{n}(\delta + \alpha + j)e^{-x}, \qquad (3.1.12)$$

where $\delta = xD$ and $D = d/dx$. It follows therefore, from (3.1.12) and (3.1.8) that

$$H_{2n}^{\mu}(x^{1/2}) = (-1)^n\, e^x \prod_{j=1}^{n}(\delta + 1/2\,\mu - 1/2 + j)\ e^{-x} \qquad (3.1.13)$$

and from (3.1.12) and (3.1.9) we have

$$H^{\mu}_{2n+1}(x^{1/2}) = (-1)^n x^{1/2} e^x \prod_{j=1}^{n} (\theta+\mu/2+1/2+j)\, e^{-x}. \tag{3.1.14}$$

Using these operational formulae, Rai and Singh [37] proved the following two generating functions for even and odd polynomials

$$\sum_{n=0}^{\infty} H^{\mu}_{2n}(x)\frac{t^n}{n!} = (1+t)^{-\frac{\mu}{2}-\frac{1}{2}} \exp\left(\frac{x^2 t}{1+t}\right), \tag{3.1.15}$$

$$\sum_{n=0}^{\infty} H^{\mu}_{2n+1}(x)\frac{t^n}{n!} = x(1+t)^{-\frac{\mu}{2}-\frac{3}{2}} \exp\left(\frac{x^2 t}{1+t}\right). \tag{3.1.16}$$

Every orthogonal set of polynomials possesses a three term recurrence relation of a simple nature. The relations (3.1.15) and (3.1.16) help us in obtaining pure and differential recurrence relations of $H^{\mu}_{2n}(x)$ and $H^{\mu}_{2n+1}(x)$.

With the help of the following relationships involving $\theta = x\, d/dx$,

$$H^{\mu}_{2n}(x^{1/2}) = (-1)^n e^x (\theta + 1/2(\mu+1))_n\, e^{-x}$$

and

$$H^{\mu}_{2n+1}(x^{1/2}) = (-1)^n x^{1/2} e^x (\theta + \tfrac{1}{2}(\mu+3))_n\, e^{-x}.$$

Srivastava, A.N. [41] obtained the following generating functions for $H_{2n}^{\mu}(x)$ and $H_{2n+1}^{\mu}(x)$:

$$\sum_{n=0}^{\infty} \frac{t^n}{[\frac{1}{2}(\mu+1)]_n n!} H_{2n+1}^{\mu}(x) = e^{-t}\ {}_0F_1\left(-;\frac{\mu+1}{2},\ x^2 t\right),$$

$$\sum_{n=0}^{\infty} \frac{t^n}{[\frac{1}{2}(\mu+3)]_n n!} H_{2n+1}^{\mu}(x) = x\, e^{-t}\ {}_0F_1\left(-;\frac{\mu+3}{2},\ x^2 t\right).$$

For the even Hermite polynomials $H_{2n}^{\mu}(x)$, we can obtain a pure recurrence relation by differentiating (3.1.15) with respect to t. The relation is

$$H_{2n+2}^{\mu}(x) + \left(2n+\frac{\mu+1}{2}-x^2\right) H_{2n}^{\mu}(x) + n\left(n+\frac{\mu-1}{2}\right) H_{2n-2}^{\mu}(x) = 0 .$$

(3.1.17)

Similarly, differentiation of (3.1.15) with respect to x would give us the following differential recurrence relation

$$[H_{2n}^{\mu}(x)]' = 2n\, H_{2n-1}^{\mu}(x) . \qquad (3.1.18)$$

Other recurrence relations which are useful in our work are

$$x\,[H_{2n+1}^{\mu}(x)]' = (\mu+1+2n)\,x\, H_{2n}^{\mu}(x) - \mu\, H_{2n+1}^{\mu}(x), \quad (3.1.19)$$

$$x[H^{\mu}_{2n+1}(x)]' = (2x^2-\mu)\, H^{\mu}_{2n+1}(x) - 2x\, H^{\mu}_{2n+2}(x), \quad (3.1.20)$$

$$x[H^{\mu}_{2n}(x)] = H^{\mu}_{2n+1}(x) + n\, H^{\mu}_{2n-1}(x), \quad (3.1.21)$$

$$x[H^{\mu}_{2n+2}(x)] = x^2\, H^{\mu}_{2n+1}(x) + (-\mu/2-1/2)\, x\, H^{\mu}_{2n}(x)$$

$$- n\, x\, H^{\mu}_{2n}(x)\,. \quad (3.1.22)$$

Each of these results may be obtained by using (3.1.15), (3.1.16) and combining various recurrence relations.

3.2 The representation $\uparrow_{w,\mu'}$ for $H^{\mu}_{2n}(x)$

As mentioned in section 2.2, Chapter 2, $\mathcal{G}(o,1)$ essentially coincides with the Lie algebra of the local Lie group $G(o,1)$ given by

$$G(o,1) = \left\{ \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ o & e^{\tau} & b & o \\ o & o & 1 & o \\ o & o & o & 1 \end{pmatrix},\ a,b,c,\tau \in \mathbb{C} \right\}.$$

The irreducible representation $\uparrow_{w,\mu'}$ of $\mathcal{G}(o,1)$ is determined by complex constants w, μ' such that $\mu' \neq o$. The spectrum of this representation is the set

$$S = \{-w+n : n \text{ is a nonnegative integer}\}$$

and the representation space V has a basis $\{f_m\}$, $m \in S$, so that

$$J^3 f_m = m\, f_m,\ E\, f_m = \mu' f_m,\ J^+ f_m = \mu' f_{m+1}\ ,$$

$$J^- f_m = (m+w)\, f_{m-1}\ , \qquad (3.2.1)$$

The commutation relations satisfied by the operators are

$$[J^3,\ J^{\pm}\ ] = \pm\, J^{\pm},\ [J^+,\ J^-\ ] = -E,$$

$$[J^{\pm},\ E] \quad = [J^3,\ E] = o. \qquad (3.2.2)$$

We can extend the realization of $\uparrow_{w,\mu'}$ defined on V to a local multiplier representation of $\mathcal{G}(o,1)$ defined on $\mathcal{A}_2$, where $\mathcal{A}_2$ is the complex vector space of all functions of z analytic in some neighborhood of the point z = o.

Let us introduce the first order linearly independent differential operators J^3, J^+ and J^- each of the form

$$A_1(x,y)\,\frac{\partial}{\partial x} + A_2(x,y)\,\frac{\partial}{\partial y} + A_3(x,y)$$

such that

$$J^3 [y^{2n} H^{\mu}_{2n} (x)] = a_n y^{2n} H^{\mu}_{2n} (x) ,$$

$$J^- [y^{2n} H^{\mu}_{2n} (x)] = b_n y^{2n-1} H^{\mu}_{2n-1} (x) , \qquad (3.2.3)$$

$$J^+ [y^{2n} H^{\mu}_{2n} (x)] = c_n y^{2n+1} H^{\mu}_{2n+1} ,$$

where a_n, b_n, c_n are expressions in n which are independent of x and y, but not necessarily of μ. Each $A_i(x,y)$, $i = 1,2,3$, on the other hand, is an expression in x and y which is independent of n but not necessarily of μ.

Using (3.2.3) and recurrence relations (3.1.18) and (3.1.21), we get the following operators

$$J^3 = y\, \partial/\partial y \ , \ J^- = y^{-1}\, \partial/\partial x, \ J^+ = -y\, \partial/\partial x + 2\, xy ,$$

$$E = 2 \ . \qquad (3.2.4)$$

Computation relations of these operators are identical with (3.2.2). For $\mu' = 2$ and $w = o$, these operators satisfy (3.2.1) and thus have a realization $\uparrow_{o,2}$.

According to theorem 1.2.1 of chapter 1, these operators generate a Lie algebra, isomorphic to $\mathcal{G}(o,1)$, which is the algebra of generalized Lie derivatives of a multiplier representation T of G(o,1) acting on cl_2. We will compute the multiplier ν explicitly. The action of the one parameter subgroup exp $c\mathcal{J}^-$, $c \in \mathbb{C}$ of G(o,1) on cl_2 is obtained by solving the equations

$$\frac{dx}{dc} = y^{-1}, \quad \frac{dy}{dc} = 0, \quad \frac{d\nu}{dc} = 0$$

with initial conditions $x(0) = x^0 \neq 0$, $y(0) = y^0 \neq 0$, $\nu(x^0, e) = \nu(y^0, e) = 1$. Here e is the identity element of G(o,1) and exp $c\mathcal{J}^-$ is given by (Miller [31], p. 10).

$$\exp c\mathcal{J}^- = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The solutions of the differential equations is

$$x(c) = \frac{c}{y^0} + x^0, \quad y(c) = y^0 \quad \text{and} \quad \nu(c) = 1.$$

Thus if $f \in cl_2$ is analytic in a neighborhood of (x^0, y^0) then

$$[T(\exp c\mathcal{J}^-)f](x^0, y^0) = f\left(\frac{c}{y^0} + x^0, y^0\right),$$

Similarly, we obtain

$$[T(\exp\ \tau\ j^3)f](x^0,y^0) = f(x^0,\ y^0e^\tau)\ ,$$

$$[T(\exp b\ j^+)\ f](x^0,y^0) = e^{-y^{02}b^2+2x^0y^0b} f(-y^0b+x^0,y^0)\ ,$$

$$[T(\exp a\ \xi\ )f\ ](x^0,y^0) = \exp\ (2a)\ f\ (x^0,y^0)\ .$$

If $g \in G(0,1)$ has coordinates (a,b,c,τ), we have

$$g = (\exp b\ j^+)(\exp c\ j^-)\ (\exp \tau\ j^3)(\exp a\,\xi)$$

and the operator $T(g)$ acting on $f \in \mathcal{C}l_2$ is given by

$$[T(g)f](x,y) = T[(\exp bj^+)(\exp cj^-)(\exp \tau j^3)(\exp a\,\xi)f](x,y)$$

$$= [T(\exp b\ j^+)\ T(\exp c\ j^-)\ T\exp\ (\tau\ j^3)\ T(\exp a\ \xi)f](x,y)$$

$$= \exp\ (yb\ (+2x - yb)\ +\ 2a)\ f\ (\ \frac{c}{y} - yb + x,\ y\ e^\tau)\ . \tag{3.2.4}$$

Every function f in cl_2 has a unique power expansion

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n,\ a_n \in \mathbb{C},$$

convergent for all $z \in \mathbb{C}$. Thus the basis functions $f_n(x,y) = y^{2n} H^{\mu}_{2n}(x)$, $n \geq 0$, of V form an analytic basis for cl_2. With respect to this analytic basis the matrix elements $B_{lk}(g)$ are defined by

$$[T(g)f_k](x,y) = \sum_{l=0}^{\infty} B_{lk}(g)\, f_l(x,y), \quad g \in G(0,1),\ k = 0,1,2\ldots \tag{3.2.5}$$

or, from (3.2.4)

$$\exp\left(2\,xyb - y^2b^2 + 2a + 2\tau k\right) y^{2k} H^{\mu}_{2k}(c/y - yb + x)$$

$$= \sum_{l=0}^{\infty} B_{lk}(g)\, y^{2l} H^{\mu}_{2l}(x)\,. \tag{3.2.6}$$

The matrix elements $B_{lk}(g)$ are to be determined by expanding the left hand side of (3.2.6) in a power series in y and then computing the coefficient of y^{2l}. We discuss a few special cases of $B_{lk}(g)$.

(i) Put $\mu = 0$ in (3.2.6) to get

$$\exp\left(2xyb - y^2b^2\right) y^{2k} H_{2k}(-yb + x + c/y)$$

$$= 2^{2k-2l} \exp(-2a - 2\tau k) \sum_{l=0}^{\infty} B_{lk}(g)\, y^{2l} H_{2l}(x)\,. \tag{3.2.7}$$

Comparing (3.2.7) with a result in ([31], p. 106, equation (4.76)}, we find

$$B_{lk}(g) = (-c/2)^{2(k-l)} \exp(2a + 2\tau k)\, L_{2l}^{2(k-l)}(-bc)\,.$$

Putting this in (3.2.6), we get the following generating function

$$\exp(2xyb - y^2b^2)\, y^{2k}\, H_{2k}^{\mu}(c/y - by + x)$$

$$= \sum_{l=0}^{\infty} (-c/2)^{2(k-l)} L_{2l}^{2(k-l)}(-bc)\, y^{2l} H_{2l}^{\mu}(x)\,. \quad (3.2.8)$$

(ii) On taking $c = 0$, $y = 1$ in (3.2.8) and using ([31], p.88)

$$c^n L_l^n(bc)\big|_{c=0} = \begin{cases} 0 & \text{if } n > 0 \\ \dfrac{(-b)^{-n}}{(-n)!} & \text{if } n \leq 0\,, \end{cases}$$

that is

$$(-c)^{k-l} L_l^{k-l}(-bc)\big|_{c=0} = \begin{cases} 0 & \text{if } k > l \\ \dfrac{(-b)^{l-k}}{(l-k)!} & \text{if } k \leq l \end{cases}$$

and replacing l by $l+k$, we get

$$\exp(-b^2-2xb)\, H_{2k}^{\mu}(x+b) = \sum_{l=0}^{\infty} \frac{(2b)^{2l}}{(2l)!} H_{2(k+l)}^{\mu}(x)\,. (3.2.9)$$

For $k = 0$, $\mu = 0$ and $b = -y$, it becomes

$$\exp(2xy - y^2) = \sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!} H_\ell(x) , \tag{3.2.10}$$

which is a well known generating function for the Hermite polynomials.

3.3 The representation $\uparrow_{w,\mu'}$ for $H^{\mu}_{2n+1}(x)$

We will now relate $H^{\mu}_{2n+1}(x)$ to the representation theory of $L[G(o,1)]$. Here, analogous arguments will be used to obtain the following operators by using (3.2.3) and recurrence relations (3.1.19).

$$J^3 = y\,\partial/\partial y,\ J^- = y^{-1}\,\partial/\partial x + \mu\,x^{-1}\,y^{-1},$$
$$J^+ = -y^{-1}/2\,\partial/\partial x + y\,(x - \mu/2\,x^{-1}),\ E = 1 . \tag{3.3.1}$$

Commutation relations satisfied by these operators are identical with (3.2.2). For $\mu' = 1$ and $w = \mu$, these operators satisfy (3.2.1) and thus have a realization $\uparrow_{\mu,1}$

As was stated in section 3.2, these operators generate a Lie algebra, isomorphic to $\mathcal{G}(o,1)$, which is the algebra of generalized Lie derivatives of a multiplier representation T of $G(o,1)$ acting on cl_2, we will compute the multiplier ν explicity. The action of one parameter subgroup $\exp cJ^-$, $c \in \mathbb{C}$ of $G(o,1)$ on cl_2 is obtained by solving the differential equations

$$\frac{dx}{dc} = y^{-1}, \quad \frac{dy}{dc} = o, \qquad \frac{d\nu}{dc} = \nu\mu x^{-1}y^{-1}$$

with initial conditions $x(o) = x^o \neq o$, $y(o) = y^o \neq o$, $\nu(x^o,e) = \nu(y^o,e) = 1$. Here e is the identity element of $G(o,1)$ and $\exp c\, j^-$ is given by ([31], p. 10)

$$\exp c\, j^- = \begin{pmatrix} 1 & c & o & o \\ o & 1 & o & o \\ o & o & 1 & o \\ o & o & o & 1 \end{pmatrix}.$$

The solution of these differential equations is

$$x(c) = \frac{c}{y^o} + x_o\,, \qquad y(c) = y^o \quad \text{and}$$

$$\nu(c) = \frac{(c/y^o + x^o)^\mu}{x^{o\mu}}.$$

Thus if $f \in \mathcal{O}l_2$ is analytic in a neighborhood of (x^o,y^o) then

$$[T(\exp cj^-)f]\,(x^o,y^o) = \frac{(c/y^o + x^o)^\mu}{x^{o\mu}}\, f(c/y^o+x^o,y^o)$$

Similarly, we obtain

$$[T(\exp \tau\, j^3)f](x^o,y^o) = f(x^o,y^oe^\tau)$$

$$[T(\exp bj^+)f]\,(x^o,y^o) = \frac{(-y^ob/2+x^o)^\mu\, \exp(-y^{o2}b^2/4+x^oy^ob)}{x^{o\mu}}$$

$$\cdot f(-y^ob/2 + x^o,\ y^o)$$

$$[T(\exp a\xi)f]\,(x^o,y^o) = \exp\,(a)\ f(x^o,y^o)\,.$$

If $g \in G(o,1)$ has coordinates (a,b,c,τ), we have

$$g = (\exp bj^+)\ \exp\,(cj^-)\ \exp\,(\tau j^3)\ \exp\,(a\,\xi)$$

and the operator $T(g)$ acting on $f \in \mathcal{H}_2$ is given by

$$[T(g)f]\,(x,y) = T[\exp bj^+)(\exp cj^-)(\exp \tau j^3)(\exp a\xi)f](x,y)$$

$$= [T(\exp bj^+)T(\exp cj^-)T(\exp \tau j^3)T(\exp a\,\xi\,)f](x,y)$$

$$= \frac{(c/y-by/2+x)^\mu\ \exp\,(\ -y^2b^2/4\ +xyb+a)}{x^\mu}\ f(c/y-yb/2+x,ye^\tau)\,, \tag{3.3.2}$$

Since $[T(g)f](x,y)$ is analytic, it can be expressed as

$$[T(g)f_k](x,y] = \sum_{l=0}^{\infty} B_{lk}(g)\ f_l(x,y),\quad k = 0,\ 1,\ 2\ \ldots$$

valid for all $g \in G(o,1)$, or

$$\frac{(c/y-by/2+x)^{\mu} \exp(-y^2b^2/4 + xyb + a + \tau(ak+1))}{x^{\mu}}$$

$$\cdot y^{2k} H^{\mu}_{2k+1}(c/y-yb/2+x)$$

$$= \sum_{\ell=0}^{\infty} \beta_{\ell k}(g)\, y^{2\ell} H^{\mu}_{2\ell+1}(x), \qquad (3.3.3)$$

where the matrix elements $B_{\ell k}(g)$ are given by equation ([31], p. 83, eqn. (4.13))

$$B_{\ell k}(g) = \exp[a + (2k+1)\tau - \mu\pi]\, C^{2k-2\ell} L^{2(k-\ell)}_{2\ell+1}(-bC)$$

valid for all integral values of ℓ, k. Since $H^{\mu}_{2n+1}(x)$, $n \in \mathbb{C}$ is analytic in x for all nonzero values of x, the infinite series (3.3.3) converges absolutely for $\left|\frac{c}{xy} - \frac{yb}{2x}\right| < 1$. Thus our main generating function becomes

$$x^{-\mu} y^{2k} (c/y-by/2+x)^{\mu} \exp\left[\frac{-y^2b^2}{4} + xyb\right] H^{\mu}_{2k+1}(c/y-by/2+x)$$

$$= \sum_{\ell=0}^{\infty} y^2 \exp(-\mu\pi)\, C^{2k+2\ell} L^{2(k-\ell)}_{2\ell+1}(-bc)\, H^{\mu}_{2\ell+1}(x) \,. \qquad (3.3.4)$$

CHAPTER 4

K_5 AND GENERALIZED HERMITE POLYNOMIALS

4.1 Introduction

The primary tools needed to deduce generating functions for various special functions are multiplier representations of local Lie groups and representations of Lie algebras by generalized Lie derivatives. These concepts were introduced in Chapter 1 with a brief survey of classical Lie theory. Chapters 2 and 3 were devoted to the theory of the complex Lie groups with three and four dimensional Lie algebras τ_3 and $\mathcal{G}(0,1)$. Generating functions for $S_n(x)$, $H_{2n}^{\mu}(x)$ and $H_{2n+1}^{\mu}(x)$ were obtained from this analysis. In the present chapter we determine the scope of our analysis by considering a more general 5-dimensional Lie algebra k_5 [31] which has realizations by generalized Lie derivatives in two complex variables. Corresponding to this Lie algebra we will obtain generating functions of generalized even and odd Hermite polynomials $H_{2n}^{\mu}(x)$ and $H_{2n+1}^{\mu}(x)$ respectively by relating these functions to the representation theory of the Lie algebra. Miller [31] has obtained identities for the Hermite polynomials by studying K_5. Weisner [51] has also obtained identities for the Hermite functions even more general than those of Miller [31] by considering a 6-

dimensional Lie algebra which contains k_5 as a subalgebra.

Following Miller [31], k_5 is the 5-dimensional Lie algebra with basis $J^{\pm}$, J^3, ξ, $\mathcal{Z}$ and commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, \quad [J^3, \mathcal{Z}] = 0,$$

$$[J^-, J^+] = \xi, \quad [J^-, \mathcal{Z}] = 2J^+, \quad [J^+, \mathcal{Z}] = 0, \tag{4.1.1}$$

$$[J^-, \xi] = [J^3, \xi] = [\mathcal{Z}, \xi] = 0.$$

Since the 4-dimensional subalgebra of k_5 generated by $J^{\pm}$, J^3, ξ is isomorphic to $\mathcal{G}(0,1)$, the theory of k_5 is of much wider applicability than the theory of $\mathcal{G}(0,1)$ presented in Chapter 3. Thus the present chapter on k_5 may be considered as an extension of Chapter 3 which was devoted to $\mathcal{G}(0,1)$.

4.2 The representation of k_5 for $H_{2n}^{\mu}(x)$

A complex 5-dimensional Lie group with elements $g(q,a,b,c,\tau)$, $q,a,b,c,\tau \in \mathbb{C}$ and multiplication law $g(q,a,b,c,\tau)\, g(q', a', b', c', \tau')$

$$= g(q+e^{2\tau} q', a+a' + e^{\tau}cb', b+e^{\tau}b' + 2e^{2\tau} cq', c+e^{-\tau}c', \tau+\tau') \tag{4.2.1}$$

was designated as K_5 by Miller [31, p. 300]. In particular.

$g(o,o,o,o,o)$ is the identity element and the inverse of $g(q,a,b,c,\tau)$ is

$$g(-qe^{-2\tau},\ -a+bc-2c^2q,\ -be^{-\tau}+2cqe^{-\tau},\ -ce^{\tau},\ \tau).$$

This group has the 5 x 5 matrix realization (cf. [31], p. 300)

$$g(q,a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & be^{-\tau} & 2a-bc & \tau \\ o & e^{\tau} & 2qe^{-\tau} & b-2qc & o \\ o & o & e^{-\tau} & -c & o \\ o & o & o & 1 & o \\ o & o & o & o & o \end{pmatrix}, \quad (4.2.2)$$

where now the group operation is matrix multiplication, clearly, the set of all group elements with $q = o$ forms sub-group of K_5 isomorphic to $G(o,1)$. The Lie algebra of K_5 is isomorphic to k_5.

A simple computation yields

$$g(q,a,b,c,\tau) = \exp(q\mathcal{Q})\exp(a\xi)\exp(bj^+)\exp(cj^-) \cdot \exp(\tau j^3), \quad (4.2.3)$$

where the elements $j^{\pm}$, j^3, ξ, $\mathcal{Q}$ generate k_5 and satisfy the commutation relation ([33], p. 299) given by (4.1.1).

Equation (4.1.1) uniquely determines K_5 as a local Lie group. Moreover, as a global group K_5 is simply connected. Clearly, the 4-dimensional subalgebra of k_5 generated by $J^{\pm}$, J^3, $\mathcal{E}$ is isomorphic to $\mathcal{G}(0,1)$. The Lie algebra k_5 is of interest because it has realizations by differential operators in one complex variable as well as in two complex variables.

We introduce the first order linearly independent differential operators J^3, J^+, J^- and Q each of the form

$$A_1(x,y)\frac{\partial}{\partial x} + A_2(x,y)\frac{\partial}{\partial y} + A_3(x,y) \tag{4.2.4}$$

such that

$$\begin{aligned}
J^3\,[y^{2n} H^{\mu}_{2n}(x)] &= a_n\, y^{2n} H^{\mu}_{2n}(x)\,,\\
J^-\,[y^{2n} H^{\mu}_{2n}(x)] &= b_n\, y^{2n-1} H^{\mu}_{2n-1}(x)\,,\\
J^+\,[y^{2n} H^{\mu}_{2n}(x)] &= c_n\, y^{2n+1} H^{\mu}_{2n+1}(x)\,,\\
Q\,[y^{2n} H^{\mu}_{2n}(x)] &= d_n\, y^{2n+2} H^{\mu}_{2n+2}(x)\,,
\end{aligned} \tag{4.2.5}$$

where a_n, b_n, c_n and d_n are expressions in n which are independent of x and y, but not necessarily of μ. Each

$A_i(x,y)$, $i = 1, 2, 3$, on the other hand, is an expression in x and y which is independent of n but not necessarily of μ.

On using the recurrence relations (3.1.18), (3.1.21), (3.1.22) and (3.2.5), we get the following operators

$$
\begin{aligned}
J^3 &= y\,\partial/\partial y\,,\\
J^+ &= -y\,\partial/\partial x + 2xy\,,\\
J^- &= y^{-1}\,\partial/\partial x\,,\\
Q &= -y^2 x\,\partial/\partial x - y^3\,\partial/\partial y + 2y^2\left(x^2 - \frac{\mu+1}{2}\right)
\end{aligned}
\qquad (4.2.6)
$$

and

$$E = 2\,.$$

These operators satisfy the commutation relations

$$
\begin{aligned}
&[J^3, J^\pm] = \pm J^\pm,\ [J^3, Q] = 2Q,\ [J^-, J^+] = E\,,\\
&[J^-, Q] = 2J^+,\ [J^+, Q] = 0,\ [J^\pm, E] = [J^3, E]\\
&\qquad = [Q, E] = 0\,.
\end{aligned}
\qquad (4.2.7)
$$

To construct a realization $\uparrow'_{\omega,\mu}$ (cf. Miller [31], p. 301) in terms of the operators (4.2.6), we find nonzero functions $f_m(x,y) = y^{2m} Z_\mu(x)$ such that equations

$$J^3 f_m = m f_m,\ E f_m = \mu' f_m,\ Q f_m = \mu' f_{m+2}\ ,$$

$$J^+ f_m = \mu' f_{m+1},\ J^- f_m = (m+w)\ f_{m+1} \tag{4.2.8}$$

are valid for all $m \in S = \{-w+n \mid n$ is an non-negative inter$\}$. Following Miller ([11], § 4.6), we can assume $\omega = o$, $\mu' = 2$ without any loss of generality. In terms of the function $Z_m(x)$ these relations become

$$\left(-\frac{d}{dx} + 2x\right) Z_m\ (x) = 2Z_{m+1}\ (x)\ ,$$

$$\frac{d}{dx}\,[Z_m(x)] = 2m\ Z_{m-1}(x)\ , \tag{4.2.9}$$

$$\left[-\ x\,\frac{d}{dx} + 2x^2 - \mu - 1 - 2m\right]\ Z_m(x) = 2Z_{m+2}\ (x)\ .$$

If we choose $Z_m(x) = H^{\mu}_{2m}(x)$, $m \in S$, then the function $f_m\ (x,y) = y^{2m}\ Z_m(x)$ form an analytic basis for a realization of the representation $\uparrow_{o,2}$ of k_5. This representation of k_5 can be extended to a local multiplier representation of K_5 by operators $T(g)$, $g \in K_5$, on the space F of all functions analytic in the neighborhood of the point $(x^o,\ y^o) = (o,o)$.

The comparison of (4.1.1) and (4.2.7) in light of a

Theorem of Miller (cf. [31], p. 18) suggests the operators J^3, J^-, J^+, Q, ξ in (4.2.6) act as generalized Lie derivatives for the multiplier representation $T : K_5 \longrightarrow F$, where F is the vector space of all analytic functions. Now we proceed to compute the multiplier representation of K_5. The action of 1-parameter group $\exp \tau j^3$, $\exp c\, j^-$, $\exp (bj^+)$, $\exp a\, \xi$ and $\exp q\, \mathcal{Q}$ are obtained by integrating the following differential equations

$$\frac{dx}{d\tau} = 0, \qquad \frac{dy}{d\tau} = y, \qquad \frac{d\nu}{d\tau} = 0,$$

$$\frac{dx}{dc} = y^{-1}, \qquad \frac{dy}{dc} = 0, \qquad \frac{d\nu}{dc} = 0,$$

$$\frac{dx}{db} = -y, \qquad \frac{dy}{db} = 0, \qquad \frac{d\nu}{db} = 2xy\nu,$$

$$\frac{dx}{da} = 0, \qquad \frac{dy}{da} = 0, \qquad \frac{d\nu}{da} = 2\nu,$$

$$\frac{dx}{dq} = -y^2 x, \quad \frac{dy}{dq} = -y^3, \quad \frac{d\nu}{dq} = 2y^2\,(x^2-(\mu/2 + 1/2))\nu,$$

subject to conditions $x(0) = x^0$, $y(0) = y^0$, $\nu(0) = 1$, where ν is multiplier of the representation.

Hence, the values of the multiplier representations of $\exp\tau j^3$, $\exp c j^-$, $\exp b j^+$, $\exp a\xi$, $\exp q\mathcal{Q}$ are given by

$$[T(\exp \tau j^3)f](x^0,y^0) = f(x^0, y^0 e^{\tau}),$$

$$[T(\exp c j^-)f](x^0,y^0) = f\left(\frac{c+x^0y^0}{y^0}, y^0\right),$$

$$[T(\exp b j^+)f](x^0,y^0) = \exp(-y^{02}b^2 + 2x^0y^0b)\ f(-y^0b+x^0, y^0),$$

$$[T(\exp a\xi)f](x^0,y^0) = \exp(2a)\ f(x^0,y^0),$$

$$[T(\exp(q\mathcal{Q}))f](x^0,y^0) = (2qy^{02}+1)^{-\mu/2-1/2}$$

$$\exp(2qx^{02}y^{02}/(2qy^{02}+1))$$

$$f(x^0/\sqrt{2qy^{02}+1},\quad y^0/\sqrt{2qy^{02}+1}),$$

If $g \in K_5$ is given by (4.2.2), we find

$$g = \exp(q\mathcal{Q})\ \exp(a\xi)\ \exp(bj^+)\ \exp(cj^-)\ \exp(\tau j^3)$$

and $$T[(\exp q\mathcal{Q})\ \exp(a\xi)\ (\exp bj^+)(\exp cj^-)(\exp \tau j^3)f](x,y)$$

$$= [T(\exp q\mathcal{Q})\ T(\exp a\xi)\ T(\exp bj^+)\ T(\exp cj^-)\ T(\exp \tau j^3)f](x,y)$$

$$= (2qy^2+1)^{-\mu/2-1/2}\ \exp\left[\frac{2qx^2y^2-y^2b^2+2xyb}{2qy^2+1} + 2a\right]$$

$$\cdot f\left(\frac{c(2qy+y^{-1})-yb+x}{\sqrt{2qy^2+1}}, \frac{ye^{\tau}}{\sqrt{2qy^2+1}}\right)\cdot \qquad (4.2.10)$$

Since $[T(g)f](x,y)$ is analytic, it can be expressed as

$$[T(g)f_k](x,y) = \sum_{\ell=0}^{\infty} \beta_{\ell k}(g)\, f_\ell(x,y), \quad k = 0,1,2 \ldots$$

valid for all $g \in K_5$, or

$$(2qy^2+1)^{-\mu/2-1/2-k} \exp\left[\frac{2qx^2y^2-y^2b^2+2xyb}{2qy^2+1} + 2a + 2k\tau\right)$$

$$\cdot y^{2k} H^{\mu}_{2k}\left(\frac{2cqy + cy^{-1} - yb + x}{\sqrt{2qy^2+1}}\right)$$

$$= \sum_{\ell=0}^{\infty} \beta_{\ell k}(g)\, y^{2\ell}\, H^{\mu}_{2\ell}(x)\,. \qquad (4.2.11)$$

We shall mention a few special cases of (4.2.11). Corresponding to some special choices with the generating function (4.2.11), the matrix elements have the following explicit expressions :

(i) For $q = 0$, the expression given in (4.2.11) is identical with (3.2.3). In view of this our $\beta_{\ell k}(g)$ is given by

$$\beta_{\ell k}(g) = \exp(2a + 2k\tau)\, c^{2k-2\ell}\, {}^{2k-2\ell}_{2\ell}(-2bc)\,.$$

(ii) For $\mu = o$, $q = -1/2$ and $a = b = \tau = o$, it reduces to

$$(1-y^2)^{-1/2-k} \exp\left[\frac{-x^2y^2}{1-y^2}\right] H_{2k}\left(\frac{c(y^{-1}-y) \quad +x}{\sqrt{1-y^2}}\right)$$

$$= \sum_{\ell=0}^{\infty} R^{2k}_{2\ell-2k}(-1/2, \sqrt{2}c)\; 2^{-\ell+k}\; y^{-2k+2\ell} H_{2\ell}(x)\,, \qquad (4.2.12)$$

where we haveused $\beta_{\ell k}(g)$ given by ([31], p. 311, equ. (9.37))

$$\beta_{\ell k}(g) = R^{2k}_{2\ell-2k}(-1/2, \sqrt{2}c)\cdot 2^{\ell-k},$$

$$R^{k}_{\ell-k}(-1/2, \sqrt{2}c) = c^{\,k-\ell}\; k!\sum_{j} \frac{(-c^2/2)^j}{j!\,(2j+k-\ell)!\,(\ell-2j)!}\,.$$

(iii) For $\mu = o$, $q = -1/2$ and $a = c = \tau = o$, (4.2.11) reduces to

$$(1-y^2)^{-1/2-k} \exp\left(\frac{-x^2y^2 - y^2b^2 + 2xyb}{1-y^2}\right) H_{2k}\left(\frac{-yb+x}{\sqrt{1-y^2}}\right)$$

$$= \sum_{\ell=k}^{\infty} \frac{1}{(2\ell-2k)!} H_{2\ell-2k}(b)\, y^{2\ell-2k}\, 2^{2k-2\ell} H_{2\ell}(x)\,, \qquad (4.2.13)$$

where we have used $\beta_{\ell k}(g)$ given by ([31], p. 307, equation (9.23))

$$\beta_{\ell k}(g) = \frac{1}{(2\ell-2k)!} H_{2\ell-2k}(b) . \qquad (4.2.14)$$

By the notational changes $k \longrightarrow m$, $\ell \rightarrow \ell+m$ (4.2.13) reduces to a known result of Miller ([31], p. 307, eqn. (9.23)) for even Hermite polynomials

$$(1-t^2)^{-m-1/2} \exp\left[\frac{2xtb-(x^2+b^2)t^2}{1-t^2}\right]$$

$$\cdot H_{2m}\left(\frac{x-bt}{\sqrt{1-t^2}}\right) = \sum_{n=0}^{\infty} \frac{t^{2n}\, 2^{-2n}}{(2n)!} H_{2n}(b)\, H_{2m+2n}(x) . \qquad (4.2.15)$$

4.3 The representation theory of k_5 for $H^{\mu}_{2n+1}(x)$

Our aim is now to extend the representation theory of k_5 to generalized odd Hermite polynomials. In exact analogy with the preceding section, our operators in the present case are

$$J^3 = y\, \partial/\partial y,$$

$$J^+ = -y/2\, \partial/\partial x + y(x-\mu/2x),$$

$$J^- = y^{-1}\, \partial/\partial x + \mu x^{-1} y^{-1}, \qquad (4.3.1)$$

$$C = -x/2\, y^2\, \partial/\partial x - 1/2\, y^3 \partial/\partial y + y^2(x^2-\mu/2-1/2),$$

$$E = 1 .$$

Construct a realization $\uparrow'_{w,\mu'}$ [cf. Miller ([31], p. 301)) in terms of the operators (4.3.1). We find nonzero functions $f_m(x,y) = y^{2m+1} Z_m(x)$ such that equations (4.2.8) are valid for all $m \in S = \{-w+n \mid n$ is an nonnegative integer$\}$. Following Miller ([31], § 4.6), we can assume $w = \mu$, $\mu' = 1$ without any loss of generality.

If we choose $Z_m(x) = H^{\mu}_{2m+1}(x)$, $m \in S$, then the function $f_m(x,y) = y^{2m+1} Z_m(x)$ form an analytic basis for a realization of the representation $\uparrow'_{\mu,1}$ of k_5. This representation of k_5 can be extended to a local multiplier representation of K_5 by operators $T(g)$, $g \in K_5$, on the space F of all functions analytic in the neighborhood of the point $(x^0,y^0) = (0,0)$.

The operators J^+, J^-, J^3, E and Q satisfy the commutation relations (4.2.7). The comparison of (4.1.1) and (4.2.7) in the light of a Theorem of Miller (cf. [31], p. 18) suggests the operators J^3, J^-, J^+, Q, E in (4.3.1) act as a generalized Lie derivatives for the multiplier representation $T : K_5 \longrightarrow F$, where F is the vector space of all analytic functions. Now we proceed to compute the multiplier representation of K_5. The action of 1-parameter group $\exp \tau J^3$, $\exp cJ^-$, $\exp bJ^+$, $\exp a\mathcal{E}$ and $\exp(q\mathcal{Q})$ are obtained by integrating the following differential equations.

$$\frac{dx}{d\tau} = o, \quad \frac{dy}{d\tau} = y, \quad \frac{d\nu}{d\tau} = o\ ;$$

$$\frac{dx}{dc} = y^{-1}, \quad \frac{dy}{dc} = o, \quad \frac{d\nu}{dc} = \nu\mu x^{-1}y^{-1}\ ,$$

$$\frac{dx}{db} = -y/2, \quad \frac{dy}{db} = o, \quad \frac{d\nu}{dt} = \nu y\ (x-\mu/2x), \qquad (4.3.2)$$

$$\frac{dx}{da} = o, \quad \frac{dy}{da} = o, \quad \frac{d\nu}{da} = \nu\ ;$$

$$\frac{dx}{dq} = -xy^2/2, \quad \frac{dy}{dq} = -y^3/2, \quad \frac{d\nu}{dq} = \nu y^2\ (x^2-\mu/2-1/2)$$

subject to conditions $x(o) = x^0$, $y(o) = y^0$, $\nu(o) = 1$, where ν is multiplier of the representation.

Hence, the values of the multiplier representations of $\exp\tau j^3$, $\exp cj^-$, $\exp bj^+$, $\exp a\zeta$ and $\exp q\mathcal{E}$ are given by

$$[T(\exp\tau j^3)f]\ (x^0,y^0) = f(x^0,y^0e^{\tau})\ ,$$

$$[T(\exp cj^-)f]\ (x^0,y^0) = \frac{(y^{0^{-1}}c+x^0)^{\mu}}{x^{\mu}}\ f(y^{0^{-1}}c + x^0,y^0)\ ,$$

$$[T(\exp bj^+)f]\ (x^0,y^0) = \exp(-(y^0b/2)^2 + x^0y^0b)$$

$$\cdot\ x^{0^{-\mu}}(-y^0b/2+x^0)^{\mu}\ f(-y^0b/2+x^0,y^0),$$

$$[T(\exp a\zeta)f]\ (x^0,y^0) = \exp(a)\ f(x^0,y^0)\ ,$$

$$[T(\exp q\Omega)f]\,(x^0,y^0) = (2qy^{02}+1)^{-\mu/2-1/2}$$

$$\exp(2qx^{02}y^{02}/(2qy^{02}+1))\, f(x^0/\sqrt{2qy^{02}+1},\, y^0/\sqrt{2qy^{02}+1})\,.$$

If $g \in K_5$ is given by(4.2.2), we find

$$g = \exp(q\Omega)\,\exp(a\mathcal{E})\,\exp(bj^+)\,\exp(cj^-)\,\exp(\tau j^3)$$

$$T[(\exp q\Omega)(\exp a\mathcal{E})\,(\exp bj^+)\,(\exp cj^-)(\exp \tau j^3)f](x,y)$$

$$= [T(\exp q\Omega)\; T(\exp a\mathcal{E})\; T(\exp bj^+)\; T(\exp cj^-)\; T(\exp \tau j^3)f](x,y)$$

$$= \frac{(c/y-by/2+x)^\mu}{x^\mu}\,(2qy^2+1)^{-(\frac{\mu+1}{2})}\;\exp[-y^2b^2/4+xyb+a]$$

$$\cdot\;\exp\,[2q(c/y-yb/2+x)^2y^2/(2qy^2+1)]$$

$$f\left(\frac{c/y-yb/2+x}{\sqrt{2qy^2+1}},\; y\,e^\tau/\sqrt{2qy^2+1}\right). \qquad (4.3.3)$$

Since $[T(g)f_k](x,y) = \sum_{l=0}^{\infty} \beta_{lk}(g)\, f_l(x,y),\ k = 0,1,2\ \ldots.$

valid for all $g \in K_5$, or

$$(c/xy-by/2x+1)^{\mu}\,(2qy^2+1)^{-(\frac{\mu+1}{2})\,-k-1/2}$$

$$\cdot\exp\left[-\frac{y^2b^2}{4}+xyb+a+(2k+1)\,\tau\right]$$

$$\cdot\exp\left[2q\,(c/y-yb/2+x)^2y^2/(2qy^2+1)\right]\,y^{2k+1}$$

$$\cdot H^{\mu}_{2k+1}\left(\frac{c/y-yb/2+x}{\sqrt{2qy^2+1}}\right)$$

$$=\sum_{\ell=0}^{\infty}\beta_{\ell k}(g)\,y^{2\ell+1}\,H^{\mu}_{2\ell+1}(x). \qquad (4.3.4)$$

It is clear from (4.2.1) and (4.2.2) that the set of all group elements with $q = o$ forms a subgroup of K_5 isomorphic to $G(o,1)$. Therefore equation (4.3.4) would yield (3.3.3) for $q = o$. In this way, we are led naturally to an interesting generalization of generating functions of $H^{\mu}_{2\ell+1}(x)$ which were considered in section 3.3. Several other results of $H^{\mu}_{2\ell+1}(x)$ can also be obtained by taking suitable values of a,b,c,τ and q.

CHAPTER 5

SL(2) AND GENERALIZED HERMITE LAGUERRE POLYNOMIALS

5.1 Introduction

Ismail, M. introduced a method to obtain all generating functions of Boas and Buck type for any given orthogonal polynomials in [21]. He applied this method to obtain the following theorem ([21], p. 205).

Theorem

The only generating function of Boas and Buck type for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x+1)$, Hermite polynomials $H_n(x)$, Laguerre polynomials $L_n^{\alpha}(x)$ and the Bessel polynomials $\phi_n\,(c,x)$ are

$$\sum_{n=o}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}\,(1+x)\; t^n \tag{5.1.1}$$

$$= (1-t)^{-\alpha-\beta-1}\,{}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \alpha+1; \frac{2xt}{(1-t)^2}\right),$$

$$\sum_{n=o}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} \left(\frac{\alpha+\beta+1+2n}{\alpha+\beta+1} \right) t^n\, P_n^{(\alpha,\beta)}\,(x+1) \tag{5.1.2}$$

$$= \frac{1+t}{(1-t)^{\alpha+\beta+2}}\; {}_2F_1\left(\frac{\alpha+\beta+3}{2}, \frac{\alpha+\beta+2}{2}; \alpha+1, \frac{2xt}{(1-t)^2}\right),$$

with two corresponding limiting cases when $\alpha+\beta+1 \longrightarrow o$,

$$\lambda_o \sum_{n=o}^{\infty} \frac{H_{2n}(x)t^{2n}}{(2n)!} + \lambda_1 \sum_{n=o}^{\infty} \frac{H_{2n+1}(x)}{(2n+1)!} t^{2n+1}$$

$$= e^{-t^2} [\lambda_o \cosh 2xt + \lambda_1 \sinh 2xt] \;, \qquad (5.1.3)$$

$$\lambda_o \sum_{n=o}^{\infty} \frac{(c)_n}{(2n)!} H_{2n}(x)\; t^{2n} + \lambda_1 \sum_{n=o}^{\infty} \frac{(c+1/2)_n}{(2n+1)!} H_{2n+1}(x) t^{2n+1}$$

$$= (1+t^2)^{-c} \{ \lambda_o \; {}_1F_1 \left(c;\ 1/2;\ \frac{x^2t^2}{t^2+1}\right) + \lambda_1 \frac{tx}{\sqrt{1+t^2}}$$

$$\cdot {}_1F_1 \left(c+1/2;\ 3/2;\ \frac{x^2t^2}{t^2+1}\right) \}, \qquad (5.1.4)$$

$$\sum_{n=o}^{\infty} \frac{(c)_n}{(\alpha+1)_n} L_n^{\alpha}(x)\; t^n = (1-t)^{-c}\; {}_1F_1 \left(c;\ \alpha+1;\ \frac{-xt}{1-t}\right), \qquad (5.1.5)$$

$$\sum_{n=o}^{\infty} L_n^{\alpha}(x) \frac{t^n}{(\alpha+1)_n} = e^t \; {}_0F_1 (-;\ 1+\alpha;\ -xt), \qquad (5.1.6)$$

$$\sum_{n=0}^{\infty} \phi_n (c,x)\; t^n = (1-t)^{-c}\; {}_2F_0 \left(\tfrac{1}{2}c;\ \tfrac{1}{2}c+1/2;\ -;\ \frac{-4xt}{(1-t)^2} \right) \;, \qquad (5.1.7)$$

$$\sum_{n=0}^{\infty} (2n+c)\, \phi_n(c,x)\, t^n = \frac{c(1+t)}{(1-t)^{c+1}}\, {}_2F_0\left(1/2\cdot c+1,\ 1/2\cdot c+1/2;\ -;\ \frac{-4xt}{(1-t)^2}\right). \qquad (5.1.8)$$

The class of generating functions (5.1.4) is contained in Brafman's class of peculiar generating function [4]. For other generating functions, see [38].

Given the generating function [38]

$$(1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)\, t^n \qquad (5.1.9)$$

for the well known associated Laguerre polynomials $L_n^{\alpha}(x)$, a natural extension of (5.1.9) is

$$(\gamma-\beta t)^{-m} \exp\left(\frac{-\alpha xt}{\gamma-\beta t}\right) = \sum_{n=0}^{\infty} L_{\alpha,\beta,\gamma,m,n}(x)\, t^n, \quad (5.1.10)$$

where n is a positive integer and the other parameters are unrestricted in general. The polynomial $L_{\alpha,\beta,\gamma,m,n}(x)$ may be regarded as generalized Hermite-Laguerre polynomials. For $m = \alpha+1$, $\alpha = \beta = \gamma = 1$, (5.1.10) reduces to (5.1.9). More importantly $L_{\alpha,\beta,\gamma,m,n}(x)$ contains the modified Laguerre polynomials $L_{\alpha,\beta,m,n}(x)$ of Goyal [15a], when $\gamma = 1$.

The generalized Hermite-Laguerre polynomials

$L_{\alpha,\beta,\gamma,m,n}(x)$ possess the following two generating functions

$$\sum_{n=0}^{\infty} \frac{(d)_n}{(m)_n} L_{\alpha,\beta,\gamma,m,n}(x)\, t^n = \frac{\gamma^{d-m}}{(\gamma-t\beta)^d}\, {}_1F_1\left(d;\, m;\, -\frac{\alpha x t}{\gamma-\beta t}\right), \tag{5.1.11}$$

$$\sum_{n=0}^{\infty} \frac{1}{(m)_n} L_{\alpha,\beta,\gamma,m,n}(x)\, t^n = \gamma^{-m}\, e^{\beta t/\gamma}\, {}_0F_1\left(\begin{matrix} - \\ m \end{matrix};\, -\frac{\alpha x t}{\gamma}\right). \tag{5.1.12}$$

The generating functions for the Hermite and the Laguerre polynomials (5.1.3) to (5.1.6) are special cases of (5.1.11) and (5.1.12).

If we substitute $\alpha = \beta = \gamma = 1$, $m = \alpha+1$ in (5.1.12) and use the relation $L_{1,1,1,\alpha+1,n}(x) = L_n^{\alpha}(x)$, we obtain the following generating function of $L_n^{\alpha}(x)$ in terms of Bessel function ([2], p. 32)

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^n}{(n+\alpha)!} = (xt)^{\alpha/2}\, e^t\, J_\alpha(2\sqrt{xt}) . \tag{5.1.13}$$

(5.1.13) shows that the reverse of $L_n^{\alpha}(x)$ are generated by

$$\sum_{n=0}^{\infty} x^n L_n^{\alpha}(1/x)\, t^n/(n+\alpha)! = e^{xt}\, t^{-\alpha/2}\, J_\alpha(2\sqrt{t}) \tag{5.1.14}$$

Thus the reversed Laguerre polynomials are Appell polynomials corresponding to $t^{-\alpha/2}\, J_\alpha\,(2\sqrt{t})$.

On differentiating (5.1.10) with respect to t and x and by comparing the coefficients of t, we get the following pure and differential recurrence relations for $L_{\alpha,\beta,\gamma,m,n}(x)$

$$\gamma^2(n+1)\,\ell_{n+1}(x) + \beta^2(m+n-1)\,\ell_{n-1}(x)$$

$$= [2\,\beta\gamma n + m\beta\gamma - \alpha x\gamma]\,\ell_n(x)\ , \tag{5.1.15}$$

$$\gamma\,\ell_n'(x) = \beta\,\ell_{n-1}'(x) - \alpha\,\ell_{n-1}(x)\ , \tag{5.1.16}$$

$$x\beta\,\ell_n'(x) + [\beta(m+n) - \alpha x]\,\ell_n(x) - \gamma(n+1)\,\ell_{n+1}(x) \equiv 0, \tag{5.1.17}$$

$$\gamma x\,\ell_n'(x) - \gamma n\,\ell_n(x) + (m+n-1)\,\beta\,\ell_{n-1}(x) = 0\,. \tag{5.1.18}$$

Here for the sake of brevity, we have used

$$L_{\alpha,\beta,\gamma,m,n}(x) = \ell_n(x)$$

and
$$\frac{d}{dx}\,[\,\ell_n(x)] = \ell_n'(x)\,.$$

The differential equation for $L_{\alpha,\beta,\gamma,m,n}(x)$ is given by

$$\left[x \frac{d^2}{dx^2} + \left(m - \frac{\alpha x}{\beta}\right) \frac{d}{dx} + \frac{\alpha}{\beta} n \right] \mathcal{U}(x) = 0 \,,$$

where

$$\mathcal{U}(x) = L_{\alpha,\beta,\gamma,m,n}(x) \,.$$

By equation (5.1.10), we have on comparison with ([15a], p. 263 eqn. (1))

$$L_{\alpha,\beta,\gamma,m,n}(x) = \gamma^{-m-n} L_{\alpha,\beta,m,n}(x) \,,$$

where $L_{\alpha,\beta,m,n}(x)$ is modified Laguerre polynomial defined by Goyal [15a].

(5.1.10) would also yield the following finite sum for $L_{\alpha,\beta,\gamma,m,n}(x)$

$$L_{\alpha,\beta,\gamma,m,n}(x) = \sum_{\gamma=0}^{n} \frac{(-1)^{\gamma}(\alpha x/\beta)^{\gamma}\beta^{n}(m)_n}{\gamma!\; \gamma^{m+n}(n-\gamma)!\; (m)_{\gamma}} \,,$$

which is a generalization of ([38], p. 203(1)) and ([15a], p. 263, eqn. (2)).

Special cases of $L_{\alpha,\beta,\gamma,m,n}(x)$ are numerous and well-known. For convenience, important special cases are given in the following table.

(a)	$L_{1,1,1,\alpha+1,n}(x)$	$L_n^{(\alpha)}(x)$	associated Laguerre polynomials [38]
(b)	$L_{1,1,1,1/2,n}(x^2)$	$(-1)^n H_{2n}(x)/2^{2n} n!$	even Hermite polynomials [38]
(c)	$L_{1,1,1,3/2,n}(x^2)$	$(-1)^n H_{2n+1}(x)/2^{2n} n!$	odd Hermite polynomials [38]
(d)	$L_{1,1,1,\frac{\mu+1}{2},n}(x^2)$	$(-1)^n H_{2n}^{\mu}(x)/n!$	generalized even Hermite polynomials [46], [10]
(e)	$L_{1,1,1,\frac{\mu+3}{2},n}(x^2)$	$(-1)^n H_{2n+1}^{\mu}(x)/n!$	generalized odd Hermite polynomials [46], [10]
(f)	$L_{-1,4u,1,\nu+1/2,n}(x^2)$	$P_{n,\nu}(x,u)/n!$	generalized Heat polynomials [5], [18]
(g)	$L_{1,1,1,-2n,n}(x)$	$(-2)^n S_n(x/2)$	Schultz-Piszachich polynomials [40]
(h)	$L_{1,1,1,-2n,n}(x)$	$(-1)^n (x)^n y_n(2/x)/n!$	Bessel polynomials [38]
(i)	$L_{1,1,1,1-\alpha,n}(x)$	$\phi_n(x, \alpha-2n)$	Bessel polynomials [27]
(j)	$L_{1,1,1,a+n-1,n}$	$R_n(a,x)$	Shively's pseudo-Laguerre polynomials [38]

Two rather pretty identities involving the Hermite and Laguerre polynomials are ([13] § 10.13 (38), 10.12 (41))

$$2^{-n/2} H_n(x+y) = \sum_{k=0}^{\infty} \binom{n}{k} H_k(\sqrt{2}/x)\, H_{n-k}(\sqrt{2}/y), \tag{5.1.19}$$

$$L_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} L_k^{\alpha}(x)\, L_{n-k}^{\beta}(y) . \tag{5.1.20}$$

The first expresses an orthogonal polynomials as a convolution of members of the same set, while the second expresses an orthogonal polynomials as a convolution of members of two different sets of orthogonal polynomials.

From (5.1.10) it follows by the usual method ([38], p. 209) that

$$L_{\alpha,\beta,\gamma,m+k,n}(x+y) = \sum_{\lambda=0}^{n} L_{\alpha,\beta,\gamma,m,\lambda}(x)\, L_{\alpha,\beta,\gamma,k,n-\lambda}(y), \tag{5.1.21}$$

which is a generalization of (5.1.20).

The first significant advance for obtaining generating functions by Lie theoretic approach was made by Weisner [50], [51], [52], who exhibits the use of this method for hypergeometric, Hermite and Bessel functions. Miller [31], [32] extended Weisner's theory further by relating it to the factorization method of Infeld and Hull [20]. Following Weisner and Miller, Jain [22], and Jain and Manocha [23] obtained generating functions corresponding to the Lie algebras sl(2) and L(S(0,1)) and thus derived a generating function [44]

$$(1+by/d)^{-\alpha-1} \exp\left(\frac{y(1+bdx)}{d(by+d)}\right) {}_0F_1\left(\begin{matrix}-\\ \alpha+1;\end{matrix}\ \frac{-xy}{(by+d)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}\left(\frac{1}{bd}\right)\left(-\frac{by}{d}\right)^n, \quad \left|\frac{by}{d}\right| < 1 . \tag{5.1.22}$$

For $b = -d = i/\sqrt{\omega}$, $i = \sqrt{-1}$, (5.1.22) yields the well known Hille-Hardy formula

$$(1-y)^{-\alpha-1} \exp\left(\frac{-(x+w)y}{1-y}\right) {}_0F_1\left(1+\alpha\ ;\ \frac{wxy}{(1-y)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(w)\, y^n, \quad |y| < 1 . \tag{5.1.23}$$

In the following sections we study Lie algebraic characterization of our generalized Hermite-Laguerre polynomials. These polynomials are related to the representation theory of sl(2). Besides treating the general polynomials $L_{\alpha,\beta,\gamma,m,n}(x)$, we shall also consider certain special cases : the Laguerre polynomials, Hermite and generalized Hermite polynomials, Heat and generalized Heat polynomials and Bessel polynomials. These special cases lead naturally to many new and known generating functions, the best known of which is the Hille-Hardy formula (5.1.23). Additional examples of the applications and their special cases may be obtained by using the table of special cases of $L_{\gamma,\beta,\gamma,m,n}(x)$.

5.2 Multiplier Representation of SL(2)

In this section we construct the multiplier representation of SL(2) in space of analytic functions $\mathcal{F}$ (cf. [31], p. 7) for the polynomial $L_{\alpha,\beta,\gamma,m,n}(x)$. Using recurrence relations (5.1.18) and (5.1.17), we get the following operators

$$J^3 = y\,\partial/\partial y + m/2,\; J^- = \frac{\alpha x y^{-1}}{\beta}\,\partial/\partial x - \alpha/\beta\,\partial/\partial y,$$

$$J^+ = \frac{\beta xy}{\alpha}\,\partial/\partial x + \frac{y^2\beta}{\alpha}\,\partial/\partial y + (m - \frac{\alpha x}{\beta})\,\frac{y\beta}{\alpha}\ .$$

These operators satisfy the relations

$$[J^3, J^{\pm}] = \pm J^{\pm},\; [J^+, J^-] = 2J^3. \qquad (5.2.1)$$

The relations (5.2.1) show that the differential operators J^3, J^+ and J^- act as generalised Lie-derivatives for the multiplier representation $T_g : \mathcal{F} \longrightarrow \mathcal{F}$, $g \in SL(2)$ and $\mathcal{F}$ be the vector space of analytic functions defined in the neighborhood N of $o \in \mathcal{C}$ (cf. [31], p. 2]. The actions of the 1-parameter subgroups $\exp rj^3$, $\exp sj^-$ and $\exp tj^+$ are obtained by integrating the differential equations (cf. [31], p. 18).

$$\frac{dx}{dr} = 0\,, \qquad \frac{dy}{dr} = y\,, \qquad \frac{d\nu}{dr} = \frac{\nu m}{2}\,,$$

$$\frac{dx}{ds} = \frac{\gamma}{\beta}\, xy^{-1}, \qquad \frac{dy}{ds} = -\frac{\gamma}{\beta}\,, \qquad \frac{d\nu}{ds} = 0\,, \tag{5.2.2}$$

and

$$\frac{dx}{dt} = \beta/\gamma\ xy, \qquad \frac{dy}{dt} = y^2\beta/\gamma, \qquad \frac{d\nu}{dt} = \left(m - \frac{\alpha x}{\beta}\right)\frac{\nu y\beta}{\gamma}\,.$$

Thus, since $\exp : sl(2) \longrightarrow SL(2)$, we see that

$$[T(\exp rj^3)f](x^0,y^0) = \exp\left(\frac{mr}{2}\right) f(x^0,\, y^0 e^r)$$

$$[T(\exp sj^-)f](x^0,y^0) = f\left(\frac{x^0}{(1-\gamma s/\beta y^0)},\, y^0(1-\gamma s/\beta y^0)\right) \tag{5.2.3}$$

and

$$[T(\exp tj^+)f](x^0,y^0) = (1-\beta y^0 t/\gamma)^{-m}$$

$$\cdot \exp\left[\frac{-\alpha x^0 y^0 \beta t/\gamma}{\beta(1-\beta y^0 t/\gamma)}\right] f\left(\frac{x^0}{(1-\beta y^0 t/\gamma)},\, \frac{y^0}{(1-\beta y^0 t/\gamma)}\right).$$

Since, in the neighborhood of identity, every $g \in SL(2)$

can be expressed as

$$g = \exp tj^{+} \cdot \exp sj^{-} \cdot \exp rj^{3},$$

we get for this g,

$$[T(g)f](x,y) = T[(\exp tj^{+})(\exp sj^{-})(\exp rj^{3})f](x,y)$$

$$= [T(\exp tj^{+}) \cdot T(\exp sj^{-})\; T(\exp rj^{3})f](x,y)\;.$$

This, in view of (5.2.3), gives

$$[T(g)f](x,y) = (1-\beta yt/\gamma)^{-m} \exp\left[\frac{-\alpha\beta xyt/\gamma}{\beta(1-\beta yt/\gamma)} + \frac{mr}{2}\right]$$

$$\cdot f\left[\frac{\beta xy}{\beta y(1-\beta yt/\gamma)-\gamma s(1-\beta yt/\gamma)^{2}}\,,\;\frac{e^{r}[y\beta-\gamma s(1-\beta yt/\gamma)]}{\beta(1-\beta yt/\gamma)}\right].$$

The complex parameters s, t and r are related to $g \in SL(2)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. Now by ([31], p. 8), we have

$$d^{-1} = e^{\gamma/2}, \quad s = -cd \quad \text{and} \quad t = -b/d.$$

Therefore,

$$[T(g)f](x,y) = (1+\beta yb/d\gamma)^{-m} \exp\left[\frac{\alpha xyb/d\gamma}{1+\beta yb/d\gamma}\right] d^{-m}$$

$$f\left[\frac{\beta xy}{[\beta y(1+\beta yb/d\gamma)+cd\gamma(1+\beta yb/d\gamma)^2]}, \frac{y\beta+cd\gamma(1+\beta yb/d\gamma)}{d^2\beta(1+\beta yb/\gamma d)}\right], \tag{5.2.4}$$

where $ad - bc = 1$.

(5.2.4) is the form of multiplier representation $T_g : \mathfrak{F} \to \mathfrak{F}$, $g \in SL(2)$,

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

lies sufficiently close to the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of $SL(2)$.

The Casimir operator in this set is given by [cf. Miller [31], p. 52]

$$C = J^+J^- + J^3J^3 - J^3 \tag{5.2.5}$$

and this operator commutes with J^+, J^- and J^3. Equation (5.2.5) enables us to rewrite the partial differential equation for $L_{\alpha,\beta,\gamma,m,n}(x)$ and help us in obtaining generating functions.

5.3 Generating Functions of $L_{\alpha,\beta,\gamma,m,n}(x)$

A set of generating functions for these polynomials is easily found. The following cases arise.

<u>Case 1</u>

We shall assume that f is common eigenfunction of the operators $\mathcal{C}$ and J^3, specifically we shall assume

$$\mathcal{C} f(x,y) = \left(\frac{m^2}{4} - \frac{m}{2} \right) f(x,y) , \tag{5.3.1}$$

$$J^3 f(x,y) = \left(\nu + \frac{m}{2} \right) f(x,y) . \tag{5.3.2}$$

In fact (5.3.1) and (5.3.2) mean that f is a common solution of the following simultaneous partial differential equations

$$\left(x\, \partial^2/\partial x^2 + \left(m - \frac{\alpha x}{\beta} \right) \partial/\partial x + \frac{\alpha y}{\beta} \partial/\partial y \right) f(x,y) = 0 , \tag{5.3.3}$$

$$(y\, \partial/\partial y - \nu)\, f(x,y) = 0 , \tag{5.3.4}$$

Hence the common solution of (5.3.3) and (5.3.4) is

$$f(x,y) = y^{\nu}\, L_{\alpha,\beta,\gamma,m,\nu}(x).$$

Thus, (5.2.4) takes the form

$$[T(g)f](x,y) = d^{-m-\nu}y^{\nu}a^{\nu}(1+\beta yb/d\gamma)^{-(m+\nu)}(1+\frac{c\gamma}{\beta ay})^{\nu}$$

$$\cdot \exp(\frac{\alpha xyb/d\gamma}{1+\beta yb/d\gamma})\, L_{\alpha,\beta,\gamma,m,\nu}(\frac{\beta xy}{(d+\beta yb/\gamma)(a\beta y+c\gamma)}),$$

(5.3.5)

where $ad - bc = 1$.

Since $[T(g)f](x,y)$ is analytic, it can be expressed as

$$[T(g)f](x,y) = \sum_{n=-\infty}^{\infty} j_n(g)\, y^{n+\nu} L_{\alpha,\beta,\gamma,m,n+\nu}(x)$$

or

$$d^{-m-\nu}\, a^{\nu}(1+\beta yb/d\gamma)^{-(m+\nu)}(1+\frac{c\gamma}{\beta ay})^{\nu} \exp(\frac{\alpha xyb/d\gamma}{1+\beta yb/d\gamma})$$

$$\cdot L_{\alpha,\beta,\gamma,m,\nu}(\frac{\beta xy}{(d+\beta yb/\gamma)(a\beta y+c\gamma)})$$

$$= \sum_{n=-\infty}^{\infty} j_n(g)\, L_{\alpha,\beta,\gamma,m,n+\nu}(x)\, y^n \qquad (5.3.6)$$

To determine $j_n(g)$, we set $x = o$ in (5.3.6), and we have

$$d^{-m-\nu}a^{\nu}(1+\beta yb/d\gamma)^{-(m+\nu)}(1+\frac{c\gamma}{\beta ay})^{\nu} L_{\alpha,\beta,\gamma,m,\nu}(o)$$

$$= \sum_{n=-\infty}^{\infty} j_n(g)\, L_{\alpha,\beta,\gamma,m,n+\nu}(o)\, y^n \qquad (5.3.7)$$

Now using the result

$$L_{\alpha,\beta,\gamma,m,n}(o) = \frac{(m)_n\,(\beta)^n}{\gamma^{m+n}\,n!}\,,$$

([44], p. 325, eq. (9)) and comparing the coefficient of y^n, we get

$$J_n(g) = \frac{d^{-m-\nu}\,a^{\nu}\,(-b/d)^n\,(\nu+1)_n\,{}_2F_1(m+\nu+n,-\nu;1+n,\,bc/ad)}{\Gamma(1+n)}\,.$$

Thus the generating function (5.3.6) becomes

$$(1+\beta yb/d\gamma)^{-(m+\nu)}\left(1+\frac{c\gamma}{\beta ay}\right)^{\nu}\exp\left[\frac{\alpha xyb/d\gamma}{(1+\beta yb/d\gamma)}\right]$$

$$\cdot L_{\alpha,\beta,\gamma,m,\nu}\,(\beta xy/(d+\beta yb/\gamma)(a\beta y+c\gamma))$$

$$= \sum_{n=-\infty}^{\infty}\frac{(1+\nu)_n}{n!}\,L_{\alpha,\beta,\gamma,m,n+\nu}(x)\,(-yb/d)^n\,{}_2F_1(m+\nu+n,-\nu;1+n;\tfrac{bc}{ad})\,,$$

(5.3.8)

where $ad - bc = 1$.

<u>Special Cases</u>

I. Setting $\gamma = 1$, $c \to o$, $ad = 1$ and replacing $\frac{yb}{d}$ by $-y$ in (5.3.8), we get a known result of Singh and Bala

([40a], p. 516, eqn. (3.2))

$$\sum_{n=0}^{\infty} \frac{(\nu+1)_n}{n!} L_{\alpha,\beta,m,\nu+n}(x)\, y^n$$

$$= (1-\beta y)^{-\nu-m} \exp\left(\frac{-\alpha xy}{1-\beta y}\right) L_{\alpha,\beta,m,\nu}\left(\frac{x}{1-\beta y}\right), \quad (5.3.9)$$

where $L_{\alpha,\beta,m,n}(x)$ is modified Laguerre polynomials of degree n, studied by Goyal [15a].

II. For $\alpha = \beta = \gamma = 1$ and $m = \alpha+1$, (5.3.8) reduces to a known generating function ([44], p. 330, eq. (29).

III. Put $\alpha = \beta = \gamma = 1$, $m = \frac{\mu+1}{2}$ and use the relation

$$L_{1,1,1,\frac{\mu+1}{2},n}(x^2) = \frac{(-1)^n H_{2n}^{\mu}(x)}{n!}$$

to get

$$(1+yb/d)^{-(\frac{\mu+1}{2}+\nu)}(1+c/ay)^{\nu} \exp\left[\frac{xyb/d}{(1+yb/d}\right] H_{2\nu}^{\mu}\left(\sqrt{\frac{xy}{(d+yb)(ay+c)}}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{H_{2n+2\nu}^{\mu}(\sqrt{x})\,(yb/d)^n}{n!}\, {}_2F_1\left(\frac{\mu+1}{2}+\nu+n, -\nu; 1+n; \frac{bc}{ad}\right),$$

Where $ad-bc=1$. (5.3.10)

Now setting $c \to 0$ and replacing by/d by y, we get a known result of Rai and Singh [37, p. 377, eqn. (12)]

IV. Put $\alpha = \beta = \gamma = 1$, $m = \frac{\mu+3}{2}$ and use the relation

$$L_{1,1,1,\frac{\mu+3}{2},n}(x^2) = \frac{(-1)^n H^{\mu}_{2n+1}(x)}{n!},$$

we get

$$(1+yb/d)^{-(\frac{\mu+3}{2}+\nu)}(1+c/ay)^{\nu} \exp\left[\frac{xyb/d}{(1+yb/d)}\right]$$

$$H^{\mu}_{2\nu+1}\left(\sqrt{\frac{xy}{(d+yb)(ay+c)}}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{H^{\mu}_{2n+2\nu+1}(x)\,(yb/d)^n}{n!}\; {}_2F_1\left(\frac{\mu+3}{2}+\nu+n; -\nu; 1+n; \frac{bc}{ad}\right),$$

Where $ad-bc=1$. (5.3.11)

Setting $c \longrightarrow o$ and replacing by/d by y, in (5.3.11), we get a known result of Rai and Singh([37], p. 378 eqn. (14))

V. Since

$$L_{-1,4u,1,\nu+\frac{1}{2},n}(x^2) = P_{n,\nu}(x,u)/n!$$

where $P_{n,\nu}(x,u)$ is the generalized Heat polynomial defined by (Haimo [18], Bragg [5])

$$P_{n,\nu}(x,u) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu+n+1/2)}{\Gamma(\nu+n-k+1/2)} x^{2n-2k} u^k,$$

(5.3.12)

a straight forward consequence of (5.3.8) is

$$(1-4uy)^{-(m+\nu+1/2)}\left(1-\frac{bc}{4(1+bc)uy}\right)^{\nu}\exp\left(\frac{xy}{1-4uy}\right)$$

$$\cdot P_{\nu,m}\left(\sqrt{\left[\frac{4uxy}{(1-4uy)[4uy-bc(1-4uy)]}\right]},\ u\right)$$

$$=\sum_{n=-\infty}^{\infty}\frac{y^n}{n!}P_{n+\nu,m}(\sqrt{x},\ u)\ {}_2F_1\left(\begin{matrix}m+\nu+n+1/2,-\nu;\\ 1+n\end{matrix};\frac{bc}{1+bc}\right),$$

(5.3.13)

where we have used ad - bc = 1 and replaced yb/d by -y.

The following result of Haimo([18], p. 757] for the generalized Heat polynomials

$$\sum_{n=0}^{\infty}P_{m+n,\nu}(x,u)\frac{t^n}{n!}=(1-4ut)^{-\nu-m-1/2}\exp\left(\frac{x^2t}{1-4ut}\right)$$

$$\cdot P_{m,\nu}\left(\frac{x}{\sqrt{1-4ut}},\ u\right),\ |ut|<1/4$$

and a result of Bragg ([5], p. 272)

$$\sum_{n=0}^{\infty}P_{n,\nu}(x,u)\frac{t^n}{n!}=(1-4ut)^{-\nu-1/2}\exp\left(\frac{x^2t}{1-4ut}\right)$$

are contained in our result (5.3.13). [See also Rosenbloom and Widder, Trans. Amer. Math. Soc. 92 (1959), p. 222, eqn. (1.2) for the special case $\nu = 0$].

Case II

We shall assume that f is common eigenfunction of the operators $\mathcal{C}$ and J^-, specifically, we shall assume

$$\mathcal{C} f(x,y) = \left(\frac{m^2}{4} - \frac{m}{2}\right) f(x,y), \tag{5.3.14}$$

$$J^- f(x,y) = -f(x,y), \tag{5.3.15}$$

that is, partial differential equations

$$\left(x\, \partial^2/\partial x^2 + \left(m - \frac{\alpha x}{\beta}\right) \partial/\partial x + \frac{\alpha y}{\beta} \partial/\partial y\right) f(x,y) = 0, \tag{5. 3.16}$$

$$\left(\frac{\alpha}{\beta} x y^{-1}\, \partial/\partial x - \frac{\alpha}{\beta} \partial/\partial y + 1\right) f(x,y) = 0 . \tag{5.3.17}$$

Assuming the general solution of (5.3.16) and (5.3.17) in the form

$$f(x,y) = e^{y\beta/\alpha} K(xy) \tag{5.3.18}$$

and substituting this in (5.3.16), we get

$$\left(u \frac{d^2}{du^2} + m \frac{d}{du} + 1\right) K(u) = 0, \quad u = xy . \tag{5.3.19}$$

This is a modification of Bessel's differential equation and has for its solution ([44], § 1.4, eqn. (11))

$$K(u) = \sqrt{m}\, u^{(-1/2)(m-1)} J_{m-1}(2\sqrt{u}) = {}_0F_1(-;\, m;\, -xy). \tag{5.3.20}$$

Thus

$$f(x,y) = e^{y\beta/\alpha}\, {}_0F_1(-;\, m;\, -xy) \tag{5.3.21}$$

and therefore

$$[T(g)f]\,(x,y) = (1+\beta yb/d\gamma)^{-m} \exp\left[\frac{\alpha xyb/d\gamma}{1+\beta yb/d\gamma}\right] d^{-m}$$

$$\cdot \exp\left[\frac{y\beta + cd\gamma\,(1+\beta yb/d\gamma)}{\alpha d^2(1+\beta yb/\gamma d)}\right] {}_0F_1\left(-,\, m;\, \frac{xy}{d^2(1+\beta yb/d\gamma)^2}\right)$$

Since $[T(g)f]\,(x,y)$ is analytic at $y = o$, it can be expanded in the form

$$[T(g)f](x,y) = \sum_{n=o}^{\infty} J_n(g)\, L_{\alpha,\beta,\gamma,m,n}(x)\, y^n$$

or

$$(1+\beta yb/d\gamma)^{-m} \exp\left[\frac{\alpha xyb/d\gamma}{1+\beta yb/d\gamma}\right] d^{-m}$$

$$\cdot \exp\left[\frac{y\beta + cd\gamma(1+\beta yb/d\gamma)}{\alpha d^2(1+\beta yb/\gamma d)}\right] {}_0F_1\left(-;\, m;\, \frac{xy}{d^2(1+\beta yb/d\gamma)^2}\right)$$

$$= \sum_{n=0}^{\infty} j_n(g)\, L_{\alpha,\beta,\gamma,m,n}(x)\, y^n . \tag{5.3.22}$$

To determine $j_n(g)$, we set $x = o$ in (5.3.22) and we have

$$d^{-m} \gamma^m \exp\left(\frac{c\gamma}{\alpha d}\right) \sum_{n=0}^{\infty} L_{\alpha,\beta,\gamma,m,n}\left(\frac{\gamma\beta}{\alpha^2 bd}\right)\left(\frac{-yb}{d}\right)^n$$

$$= \sum_{n=0}^{\infty} j_n(g)\, L_{\alpha,\beta,\gamma,m,n}(o)\, y^n .$$

By using the result (5.1.10) and comparing the coefficient of y^n, we get

$$j_n(g) = \frac{(-1)^n d^{-m-n} b^n \gamma^{2m+n} \beta^{-n} n! \exp\left(\frac{c\gamma}{\alpha d}\right) L_{\alpha,\beta,\gamma,m,n}\left(\frac{\gamma\beta}{\alpha^2 bd}\right)}{(m)_n} .$$

Thus the generating function (5.3.22) becomes

$$(1+\beta yb/\gamma d)^{-m} \exp\left[\frac{d\alpha^2 xyb/\gamma + y\beta}{d\alpha\,(d+\beta yb/\gamma)}\right] {}_0F_1\left(-;\ m;\ \frac{-xy}{(d+\beta yb/\gamma)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\gamma^{2m} n!\, L_{\alpha,\beta,\gamma,m,n}\left(\frac{\gamma\beta}{\alpha^2 bd}\right) L_{\alpha,\beta,\gamma,m,n}(x)\left(\frac{-\gamma by}{d\beta}\right)^n}{(m)_n} . \tag{5.3.23}$$

<u>Special cases</u>

I. For $\alpha = \beta = \gamma = 1$ and $m = \alpha+1$, it reduces to a known generating function ([44], p. 333, eqn. (46)).

II. Setting $b = -d = i/\sqrt{W}$ where $i = \sqrt{-1}$ and replacing by/γ by $-y$ in (5.3.23), we get

$$(1-y)^{-m} \exp\left(\frac{-y(\alpha x/\beta + w\gamma/\alpha)}{1-y} \right) {}_0F_1\left(\begin{matrix} - \, ; \\ m \, ; \end{matrix} \; \frac{+\gamma/\beta \; xyw}{(1-y)^2} \right)$$

$$= \sum_{n=0}^{\infty} \frac{\gamma^{2m+2n} \, n!}{(m)_n \beta^{2n}} \, L_{\alpha,\beta,\gamma,m,n}(x) \, L_{\alpha,\beta,\gamma,m,n}\left(\frac{\gamma\beta w}{\alpha^2} \right) y^n, \quad |y| < 1, \tag{5.3.24}$$

which gives the well known Hille-Hardy formula (5.1.23) for $\alpha = \beta = \gamma = 1$.

III. The above result (5.3.24) also reduces to a known résult of Singh and Bala [40a, p. 520, eqn (4.2)] for $\gamma = \alpha = a$ and $\beta = b$.

IV. On setting $\alpha = \beta = \gamma = 1$, $m = 1/2$, replacing x by x^2, b by b^2, d by d^2 and using the relation

$$L_{1,1,1,1/2,n}(x^2) = \frac{(-1)^n H_{2n}(x)}{2^{2n} \, n!}$$

in (5.3.23), we get

$$(1+yb^2/d^2)^{-1/2} \exp\left[\frac{b^2 d^2 x^2 y + y}{d^2(d^2+yb^2)} \right] {}_0F_1\left(- ; 1/2 ; \frac{-x^2 y}{(d^2+yb^2)^2} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (b/d)^{2n} \, 2^{-4n} \, H_{2n}(1/bd) \, H_{2n}(x) \, y^n}{n! \, (1/2)_n} \qquad (5.3.25)$$

V. Put $\alpha = \beta = \gamma = 1$, $m = \frac{\mu+1}{2}$ and use the relation

$$L_{1,1,1,\frac{\mu+1}{2},n}(x^2) = \frac{(-1)^n}{n!} H_{2n}^{\mu}(x)$$

in (5.3.23) to get

$$(1+yb/d)^{-(\frac{\mu+1}{2})} \exp\left[\frac{bdxy+y}{d(d+yb)}\right] {}_0F_1\left(-; \frac{\mu+1}{2}; \frac{-xy}{(d+yb)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{H_{2n}^{\mu}(1/bd) \, H_{2n}^{\mu}(x) \, (-by/d)^n}{n! \, (\frac{\mu+1}{2})_n} \qquad (5.3.26)$$

VI. Setting $\alpha = -1$, $\gamma = 1$, $\beta = 4u$, $m = \nu + 1/2$ in (5.3.23) and using the relation [5], [18]

$$L_{-1,\, 4u,1,\nu+1/2,n}(x^2) = P_{n,\nu}(x,u)/n!,$$

we get

$$(1+4uby/d)^{-(\nu+1/2)} \exp\left[\frac{db\,xy + 4uy}{-d(d+4uyb)}\right]$$

$${}_0F_1\left(-;\nu+1/2;\frac{-xy}{(d+4uyb)^2}\right)$$

$$= \sum_{n=o}^{\infty} \frac{(-b/4ud)^n}{n!\,(\nu+1/2)_n} P_{n,\nu}\left(\sqrt{\frac{4u}{db}},\, u\right) P_{n,\nu}\left(\sqrt{x}, u\right) y^n. \tag{5.3.27}$$

Now setting $u = -1$ in (5.3.27) and using

$$P_{n,o}(x,-1) = H_{2n}(x/2),$$

we get the following generating function for the product of generalized Heat polynomials.

$$\left(1+\frac{4by}{d}\right)^{-(\nu+1/2)} \exp\left[\frac{bd\,xy + 4y}{-d(d+4yb)}\right]$$

$${}_oF_1\left(\,;\, \nu + 1/2\,;\, \frac{-xy}{(d+4\,yb)^2}\right)$$

$$= \sum_{n=o}^{\infty} \frac{(-b/d)^n\, 2^{-2n}}{n!\,(\nu+1/2)_n} H_{2n}\left(\frac{2}{\sqrt{db}}\right) H_{2n}\left(\frac{\sqrt{x}}{2}\right) y^n. \tag{5.3.28}$$

On replacing x by $4x^2$, y by $y/4$ and taking $\nu = o$ in (5.3.28), we get (5.3.25).

REFERENCES

[1] Al-Salam, W.A. : The Bessel polynomials, Duke Math. J. 24 (1957), 529-545.

[2] Boas, R.P. and Buck, R.C. : Polynomial expansions of analytic functions, Springer-Verlag, Berlin, 1958.

[3] Bochner, S. : Über Sturm-Liouvillesche polynomsystem, Math. Z. 29 (1929), 730-736.

[4] Brafman, F. : Generating functions of Jacobi and related polynomials, Proc. Amer. Math. Soc. 4 (1951), 942-949.

[5] Bragg, L.R. : The radial heat polynomials and related functions, Trans. Amer. Math. Soc. 119 (1965), 270-290.

[6] ---- : Products of certain generalized Hermite polynomials; associated relations, Boll. Un. Mat. Ital. (4) 1 (1968), 347-355.

[7] Bragg, L.R. and Dettman, J.W. : Analogous function theories for the heat, wave, and Laplace equations. Rockey Mountain J. Math. 13 (1983), 191-214.

[8] Carlitz, L. : A note on Laguerre polynomials Michigan Math. J. 7 (1960), 219-223.

[9] Dornhoff, L. : Group representation theory, Maral Dekker, New York, 1971.

[10] Chihara, T.S. : An Introduction to orthogonal polynomials, Gordon and Breach, New York 1978.

[11] Dutta, M. Chatterjea, S.K. and More, K.L. : On a class of Generalized Hermite polynomials, Bull. Int. Math. Acad. Sinica, 3, No. 2 (1975), 277–281.

[12] Erdelyi, A. : Higher Transcendental Functions Vol. I, McGraw Hill, New York, 1953.

[13] Erdelyi, A. : Higher Transcedental Functions Vol. II, McGraw Hill, New York, 1953.

[14] Erdelyi, A. : Higher Transcendental Functions Vol. III, McGraw Hill, New York, 1953.

[15] Exton, H. : Multiple Hypergeometric Functions and applications, Ellis Horwood Ltd., Chichester, 1976.

[15a] Goyal, G.K. : Modified Laguerre polynomials, Vijnana Parishad Anusandhan Patrika, 26, No. 3 (1983), 263–266.

[16] Grosswald, E. : Bessel polynomials, Spring-Verlag, Berlin, Heidelberg and New York, 1978.

[17] Gupta, R.P. and Jain G.C. : A generalized Hermite distribution and its properties, SIAM J. Appl. Math. 27 (1974), 359-363.

[18] Haimo, D.T. : Expansions in terms of generalized heat polynomials and of their Appell transforms, J. Math. Mech. 15 (1966), 735-768.

[19] Hamza, A.M. : On generating functions of Bessel Polynomials, Bull. Fac. Engg. 2 (1974), 208-213.

[20] Infeld, L. and Hull, T. : The Factorization Method, Rev. Mod. Phys. 23 (1951), 21-68.

[21] Ismail, M.E.H. : On obtaining generating function of Boas and Buck type for orthogonal polynomials, Siam J. Math. Anal. 5 (1974), 202-208.

[22] Jain, S. : Generating functions for Laguerre polynomial, Jour. Math. Phys. Sci. 10, No. 1 (1976), 1-4.

[23] Jain, S. and Manocha, H.L. : Special linear group and generating functions, Comment. Math. Univ. St. Pauli, 26, No. 3 (1977), 105-113.

[24] Joshi, C.M. and Prajapat, M.L. : Some results concerning generalized Hermite polynomials, Indag. Math. 39, (1977), 208-217.

[25] Kalinowski, M.W. and Seweryński, M. : Differential equation for Hermite-Bell polynomials, Math. Proc. Cambridge Phil. Soc. 91 (1982), 259-265.

[26] Kaufman, B. : Special functions of Mathematical physics from the viewpoint of Lie algebra, Jour. Math. Phys. 7, No. 3, (1966), 447-457.

[27] Krall, H.L., and Frink, O. : A new class of orthogonal polynomials: The Bessel polynomials, Trans. Amer. Math. Soc. 65 (1949), 100-115.

[28] Lahiri, M. : On generalization of Hermite Polynomials, Proc. Amer. Math. Soc. 27, (1971), 117-121.

[29] Matsushima, Y. : Differentiable manifolds. Translated by E.T. Kobayashi, New York, Marcel Dekker, 1972.

[30] McBridge, E.B. : Obtaining generating functions, Vol. 21 Springer Verlag, New York, 1971.

[31] Miller, W. : Lie theory and special functions, Academic Press, New York, 1968.

[32] Miller, W. : Symmetry groups and their applications, Academic Press, New York, 1972.

[33] Miller, W. : On Lie Algebras and some special functions of Mathematical Physics, Mem. Amer. Math. Soc. No. 50 (1964).

[34] More, K.L. : A new class of generalised Hermite polynomials, Communicated to Mathematica (Cluj).

[35] Pearce, C.E.M., and Potts, R.B. : Symmetric square roots of the infinite identity matrix, J. Austral. Math. Soc. (Series A) 22, (1976), 343-361.

[36] Pontrjagin, L.S. : Topological Groups, Princeton Univ. Press, Princeton, New Jersey, 1958.

[37] Rai, P.N. and Singh, S.N. : On a class of generalized Hermite polynomials, Math. student 46, No. 4 (1978), 374-378.

[38] Rainville, E.D. : Special functions, The MacMillan Company, New York 1960 .

[39] Sagle, A.A. and Walde, R.E. : Introduction to Lie groups and Lie algebras, Academic Press, 1973.

[40] Schultz-Piszachich, W. : Eine Kasse isotroper Turbulenzfelder mit Permanenzeigenschaft und existenten Tripelkorrelationen, ZAMM 56 (1976), 303-314.

[40a] Singh, S.N. and Bala, R.N. : Group-Theoretic Origins of Certain generating functions of the Modified Laguerre polynomials, Indian J. Pure appl. Math.17,No.4(1986), 512-521.

[41] Srivastava, A.N. : Some generating relations connected with a class of generalized Hermite polynomials, J. Math. student, 49, No. 3 (1981), 249-254.

[42] Srivastava, H.M. : Some orthogonal polynomials representing the energy spectral functions for a family of isotropic turbulency fields, ZAMM 61 (1984), 255-257.

[43] Srivastava, H.M. and Karlsson, P.W. : Gaussian Hypergeometric Series, Ellis Horwood Ltd., Chichester, 1984.

[44] Srivastava, H.M. and Manocha, H.L. : A treatise on generating functions, Ellis Horwood Ltd. Chichester, 1984.

[45] Srivastava, H.M., and Singhal, J.P. : A class of polynomials defined by generalized Rodrigues formula, Ann. Mat. Pura Appl., IV, Ser. 90 (1971), 75-85.

[46] Szego, G. : Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. 23, Fourth edition, Amer. Math. Soc. Providence Rhode Island 1975.

[47] Talman, J.D. : Special Functions - A Group theoretic approach, W.A. Benjamin, Inc. New York (1968).

[48] Vilenkin, N.J. : Special Functions and the theory of Group representations, American mathematical Society providence, rhode Island, 1968.

[49] Wawrzynczyk, A. : Group Representations and special functions, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, 1984.

[50] Weisner, L. : Group theoretic origin of certain generating functions, Pacific J. Math. 5 (1955), 1033-1039.

[51] Weisner, L. : Generating Functions for Hermite Functions, Can. J. Math. 11 (1959), 141-147.

[52] Weisner, L. : Generating Functions for Bessel Functions, Can. J. Math. 11 (1959), 148-155.

[53] Werner, U., and Pietzsch, W. : Die polynome $S_n(x)$ zur Darstellung der Energiespektralfunktionen idealer Turbulenzfelder, ZAMM 58 (1978), 166-168.